Uniform RC-positivity of direct image bundles *

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Abstract

The concept of RC-positivity and uniform RC-positivity is introduced by Xiaokui Yang to solve a conjecture of Yau on projectivity and rational connectedness of a compact Kähler manifold with positive holomorphic sectional curvature. Some main theorems in Yang's proof hold under a weaker condition called weak RC-positivity. It is therefore natural to ask if (uniform) weak RC-positivity implies (uniform) RC-positivity. Another motivation for studying this problem is to understand the relation between rational connectedness of X and (uniform) RC-positivity of the holomorphic tangent bundle TX.

In this paper, we obtain results in this direction. In particular, we show that if a vector bundle E is uniformly weakly RC-positive, then $S^kE \otimes \det E$ is uniformly RC-positive for any $k \geq 0$, and S^kE is uniformly RC-positive for k large. We also discuss an approach that might lead to a solution to the question of whether weak RC-positivity of E implies RC-positivity of E.

1 Introduction

In [Yan18], Yang introduces the notion of RC-positivity as a differential geometric counterpart of rational connectedness. RC-positivity plays a crucial role in Yang's proof of a conjecture of Yau: If a compact Kähler manifold has positive holomorphic sectional curvature, then the manifold is projective and rationally connected. A stronger notion called uniform RC-positivity is introduced by Yang in [Yan20] which also can be used to prove the same conjecture of Yau. For the semipositive case of Yau's conjecture, see [HW20, Mat20, Mat22].

Let us recall the definition of RC-positivity and uniform RC-positivity. Let E be a holomorphic vector bundle of rank r over a compact complex manifold X of dimension n. Given a Hermitian metric H on E, we denote the Chern curvature of H by Θ^H, which is an End$E$-valued $(1,1)$-form. We denote by TX the holomorphic tangent bundle of X. For a vector $u \in E_t$ and a tangent vector $v \in T_tX$ with $t \in X$, we define the expression

$$H(\Theta^H u, u)(v, \bar{v})$$

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to be $\sum_{j,k} H(\Theta^H_{j\bar{k}}u,u)v_j\bar{v}_k$ locally where we write the curvature $\Theta^H = \sum_{j,k}\Theta^H_{j\bar{k}}dt_j \wedge d\bar{t}_k$ and $v = \sum_j v_j\partial/\partial t_j$.

Definition 1. A Hermitian metric H on a holomorphic vector bundle $E \to X$ is called RC-positive if for any $t \in X$ and any nonzero $u \in E_t$, there is a nonzero tangent vector $v \in T_tX$ such that $H(\Theta^H u,u)(v,\bar{v}) > 0$. On the other hand, a Hermitian metric H is called uniformly RC-positive if for any $t \in X$, there is a nonzero tangent vector $v \in T_tX$ such that for any nonzero $u \in E_t$, we have $H(\Theta^H u,u)(v,\bar{v}) > 0$.

A holomorphic vector bundle $E \to X$ is called (uniformly) RC-positive if it admits a (uniformly) RC-positive Hermitian metric.

It is clear that uniform RC-positivity implies RC-positivity. To motivate the definition of (uniform) weak RC-positivity, let us consider a Hermitian metric H on E and the induced metric h on the line bundle $O_{P(E^*)}(1)$ over the projectivized bundle $P(E^*)$. By a standard computation (for example, see [Yan18, Formula (4.5)]), we know that if (E,H) is RC-positive, then the curvature Θ of h is positive on every fiber and has at least r positive eigenvalues at every point in $P(E^*)$. The existence of such a metric h on $O_{P(E^*)}(1)$ is called weak RC-positivity of E ([Yan18, Definition 3.3]).

Similarly, using the same computation, we see that if (E,H) is uniformly RC-positive, then the curvature Θ of the induced metric h satisfies

a. Θ is positive on every fiber.

b. Θ has at least r positive eigenvalues at every point in $P(E^*)$.

c. For any point $t \in X$, there exists a nonzero tangent vector $v \in T_tX$, such that $\Theta(\tilde{v},\bar{\tilde{v}})|_{(t,[\zeta])} > 0$ for any lift $\tilde{v}$ of v to $T_{(t,[\zeta])}P(E^*)$.

Following Yang, we call the existence of such a metric h on $O_{P(E^*)}(1)$ uniform weak RC-positivity of E. Note that in the third condition, we consider the lifts to the tangent space $T_{(t,[\zeta])}P(E^*)$ for any point $[\zeta]$ in the fiber $P(E^*_t)$ not just one point $[\zeta]$. The second condition is implied by the first and the third, so we will omit it later on. Let us summarize the definition.

Definition 2. The bundle E is called weakly RC-positive if there exists a metric h on $O_{P(E^*)}(1)$ with properties a and b. The bundle E is called uniformly weakly RC-positive if there exists a metric h on $O_{P(E^*)}(1)$ with properties a and c.

In Yang's solution to Yau's conjecture, two main theorems [Yan18, Theroem 1.3 and Theorem 1.4], although formulated in terms of RC-positivity, hold under weak RC-positivity. So, it is natural to ask if weak RC-positivity of E implies RC-positivity of E ([Yan18, Question 7.11] and [Ina21, Problem 13]). This question has the same flavor as a conjecture of Griffiths [Gri69]: If E is ample, then E is Griffiths positive. For the developments on the Griffiths conjecture, see [Ume73, CF90, Ber09, MT07, LSY13, LY14, FLW20, Dem21, Pin21, Fin21, Fin22, Wu22, Wu23a, Wu23b, MZ23, Lem26, Mur26, Wu25].

In this paper, we make some progress in this direction. In particular, we prove the following theorem regarding uniform RC-positivity.

Theorem 3. *If E is uniformly weakly RC-positive over a compact Kähler manifold X, then $S^k E \otimes \det E$ is uniformly RC-positive for any $k \geq 0$, and $S^k E$ is uniformly RC-positive for k large.*

Another motivation for establishing Theorem 3 is to understand the relation between rational connectedness of X and (uniform) RC-positivity of the holomorphic tangent bundle TX. According to Yang [Yan18, Theorem 1.4] and [Yan20, Theorem 1.3], for a compact Kähler manifold X^n, if one of the following is true, then X is projective and rationally connected.

1. The holomorphic tangent bundle TX is uniformly RC-positive.
2. The exterior power $\wedge^p TX$ is RC-positive for $1 \leq p \leq n$.

One can ask if the converse is true ([Yan20, Problem 4.15]). A partial converse is proved in [Yan20, Theorem 1.4]: if X is projective and rationally connected, then the line bundle $O_{P(\wedge^p TX)}(-1)$ is RC-positive for $1 \leq p \leq n$. So, Theorem 3 can be viewed as a step towards this converse problem: constructing uniformly RC-positive Hermitian metrics out of metrics on the line bundle $O_{P(E^*)}(1)$.

We also prove a lemma (Lemma 6 in Section 4) and discuss how a variant of this lemma might lead to a solution to the original question of Yang, namely, weak RC-positivity of E implying RC positivity of E (see [Wu26b, Theorem 3] for a partial answer).

For the proof of Theorem 3, instead of the fibration $p : P(E^*) \to X$, we will work on a more general fibration and prove a general theorem which contains Theorem 3 as a special case. We consider a proper holomorphic surjection $p : \mathcal{X}^{n+m} \to Y^m$ between two complex manifolds with $\mathcal{X}$ Kähler, Y compact, and the differential dp surjective at every point. We denote the fibers $p^{-1}(t)$ by $\mathcal{X}_t$ for $t \in Y$. Let (L, h) be a Hermitian line bundle over $\mathcal{X}$. Let

$$V_t = H^0(\mathcal{X}_t, L|_{\mathcal{X}_t} \otimes K_{\mathcal{X}_t}).$$

We assume that $\dim V_t$ is independent of $t \in Y$. So, the direct image of the sheaf of sections of $L \otimes K_{\mathcal{X}/Y}$ is locally free by Grauert's direct image theorem, where $K_{\mathcal{X}/Y}$ is the relative canonical bundle. We denote by V the associated vector bundle over Y. There is a naturally defined Hermitian metric H on V. For u in V_t with $t \in Y$,

$$H(u, u) := \int_{\mathcal{X}_t} h(u, u). \tag{1}$$

We extend the metric h to act on sections u of $L|_{\mathcal{X}_t} \otimes K_{\mathcal{X}_t}$ so that $h(u, u)$ is an (n, n)-form on $\mathcal{X}_t$. In terms of local coordinates, if $u = u' \otimes e$ with u' an $(n, 0)$-form and e a frame of

$L|_{\mathcal{X}_t}$, then $h(u,u) = c_n u' \wedge \overline{u'} h(e,e)$ where $c_n = i^{n^2}$. Under this more general fibration, we can show

Theorem 4. *If the curvature Θ of h is positive on every fiber, and for any point $t \in Y$, there exists a nonzero tangent vector $v \in T_tY$, such that $\Theta(\tilde{v}, \bar{\tilde{v}})|_{(t,z)} > 0$ for any lift $\tilde{v}$ of v to $T_{(t,z)}\mathcal{X}$, then the Hermitian bundle (V, H) is uniformly RC-positive.*

Actually, the precise statement we prove in Theorem 4 is: For a fixed point $t_0 \in Y$, if the curvature Θ of h is positive on the fiber $\mathcal{X}_{t_0}$, and there exists a nonzero tangent vector $v \in T_{t_0}Y$, such that $\Theta(\tilde{v}, \bar{\tilde{v}})|_{(t_0,z)} > 0$ for any lift $\tilde{v}$ of v to $T_{(t_0,z)}\mathcal{X}$, then the Hermitian bundle (V, H) is uniformly RC-positive at t_0.

Now, we consider the fibration $p : P(E^*) \to X$, and we assume X is Kähler to make sure $P(E^*)$ is Kähler (see [Wu25, Subsection 5.2]). Therefore, by Theorem 4, we have Theorem 3. Indeed, the vector bundle V in Theorem 4 is associated with the direct image of $L \otimes K_{\mathcal{X}/Y}$. In the present situation, the relative canonical bundle $K_{P(E^*)/X}$ is isomorphic to $O_{P(E^*)}(-r) \otimes p^* \det E$. If we choose $O_{P(E^*)}(r+k)$ for the line bundle L, then V is $S^k \otimes \det E$. On the other hand, if we choose $O_{P(E^*)}(k) \otimes K^{-1}_{P(E^*)/X}$ for L, then V is $S^k E$ (we use an arbitrary metric g on $K^{-1}_{P(E^*)/X}$, and the effect of g can be absorbed by taking k large).

The proof of Theorem 4 is an adaptation of [Wu25, Section 3], but we still include the details for completeness (the original argument is due to Berndtsson in [Ber09, Section 4] and [Ber11, Section 2]. See also [CCP25]).

This paper is organized as follows. In Section 2, we give a local expression for the assumption (uniform weak RC-positivity) in Theorem 4 which will be used in the proof of the main theorem. In Section 3, we prove Theorem 4. In Section 4, we discuss a characterization of weak RC-positivity and its possible application.

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2 Preliminary

Let $p : \mathcal{X}^{m+n} \to Y^m$ be the fibration in the introduction, and $(L, h) \to \mathcal{X}$ be the Hermitian line bundle. In this section, we will explain the meaning of the assumption (uniform weak RC-positivity) in Theorem 4.

Assume that the Hermitian metric h on the line bundle L has its curvature Θ positive on every fiber. For such an h, we can decompose its curvature $\Theta = \Theta_{\mathcal{H}} + \Theta_{\mathcal{V}}$ where $\Theta_{\mathcal{H}}$ is the horizontal component and $\Theta_{\mathcal{V}}$ is the vertical component. The horizontal component $\Theta_{\mathcal{H}}$ can be viewed as an element in $C^\infty(\mathcal{X}, p^*(\wedge^{1,1}T^*Y))$.

We denote by $(t_1,\dots,t_m)$ the local coordinates in Y and by $(t_1,\dots,t_m,z_1,\dots,z_n)$ the local coordinates in $\mathcal{X}$, and assume that $h=e^{-\phi}$ locally. We write $\phi_{i\bar{j}}=\phi_{t_i\bar{t}_j}$, $\phi_{\lambda\bar{\mu}}=\phi_{z_\lambda\bar{z}_\mu}$, etc. Since Θ is positive on each fiber, the matrix $(\phi_{\lambda\bar{\mu}})$ is positive definite, and we denote its inverse by $(\phi^{\lambda\bar{\mu}})$. The horizontal component $\Theta_{\mathcal{H}}$ and the vertical component $\Theta_{\mathcal{V}}$ have the local expressions

$$\Theta_{\mathcal{H}}=\sum_{i,j}(\phi_{i\bar{j}}-\sum_{\lambda,\mu}\phi_{i\bar{\mu}}\phi^{\lambda\bar{\mu}}\phi_{\lambda\bar{j}})dt_i\wedge d\bar{t}_j \tag{2}$$

$$\Theta_{\mathcal{V}}=\sum_{\lambda,\mu}\phi_{\lambda\bar{\mu}}\delta z_\lambda\wedge\delta\bar{z}_\mu \tag{3}$$

where $\delta z_\lambda=dz_\lambda+\sum_{i,\mu}\phi^{\lambda\bar{\mu}}\phi_{i\bar{\mu}}dt_i$. That (2) and (3) are independent of local coordinates is discussed in [FLW19, Subsection 1.1], and the horizontal component $\Theta_{\mathcal{H}}$ is called the geodesic curvature there.

Recall the assumption in Theorem 4: the curvature Θ of h is positive on every fiber, and for any point $t\in Y$, there exists a nonzero tangent vector $v\in T_tY$, such that $\Theta(\tilde{v},\bar{\tilde{v}})|_{(t,z)}>0$ for any lift $\tilde{v}$ of v to $T_{(t,z)}\mathcal{X}$. We may assume the tangent vector v is $\partial/\partial t_1$ in the local coordinates above, and we denote the lift by $\tilde{v}=\partial/\partial t_1+\sum_\lambda a_\lambda\partial/\partial z_\lambda$ with $a_\lambda\in\mathbb{C}$. We then have

$$\begin{aligned}\Theta(\tilde{v},\bar{\tilde{v}})&=\phi_{1\bar{1}}+\sum_\mu\phi_{1\bar{\mu}}\overline{a_\mu}+\sum_\lambda\phi_{\lambda\bar{1}}a_\lambda+\sum_{\lambda,\mu}\phi_{\lambda\bar{\mu}}a_\lambda\overline{a_\mu}\\&=\phi_{1\bar{1}}-\sum_{\lambda,\mu}\phi_{1\bar{\mu}}\phi^{\lambda\bar{\mu}}\phi_{\lambda\bar{1}}+\sum_{\lambda,\mu}\phi_{1\bar{\mu}}\phi^{\lambda\bar{\mu}}\phi_{\lambda\bar{1}}+\sum_\mu\phi_{1\bar{\mu}}\overline{a_\mu}+\sum_\lambda\phi_{\lambda\bar{1}}a_\lambda+\sum_{\lambda,\mu}\phi_{\lambda\bar{\mu}}a_\lambda\overline{a_\mu}\\&=\phi_{1\bar{1}}-\sum_{\lambda,\mu}\phi_{1\bar{\mu}}\phi^{\lambda\bar{\mu}}\phi_{\lambda\bar{1}}+\|\sqrt{A^{-1}}\phi_1+\sqrt{A}\bar{a}\|^2,\end{aligned} \tag{4}$$

where in the last equality we denote the matrix $(\phi_{\lambda\bar{\mu}})$ by A, the column vector $(\phi_{\lambda\bar{1}})$ by ϕ_1, and the column vector (a_λ) by a. Let us summarize the computation as a lemma.

Lemma 5. $\Theta(\tilde{v},\bar{\tilde{v}})>0$ *for any lift* $\tilde{v}$ *if and only if* $\phi_{1\bar{1}}-\sum_{\lambda,\mu}\phi_{1\bar{\mu}}\phi^{\lambda\bar{\mu}}\phi_{\lambda\bar{1}}>0$*, which is also equivalent to* $\Theta_{\mathcal{H}}(\tilde{v},\bar{\tilde{v}})>0$.

3 Proof of Theorem 4

Recall the setup: $p:\mathcal{X}^{m+n}\to Y^m$ is a proper holomorphic submersion with $\mathcal{X}$ Kähler and Y compact, $(L,h)\to\mathcal{X}$ is a Hermitian line bundle, and $V\to Y$ is a holomorphic vector bundle associated with the direct image $p_*(L\otimes K_{\mathcal{X}/Y})$. The fibers of V are $V_t=H^0(\mathcal{X}_t,L|_{\mathcal{X}_t}\otimes K_{\mathcal{X}_t})$. The bundle V carries a Hermitian metric $H(u,u)=\int_{\mathcal{X}_t}h(u,u)$.

A smooth local section u of the bundle V is represented by a smooth $(n,0)$-from with values in L over $p^{-1}(W)$ for some open set W in Y such that the restriction to each fiber is holomorphic. Any representative of u is denoted by $\mathbf{u}$. We use $(t_1, \ldots, t_m)$ for local coordinates in Y. In general, we have

$$\bar{\partial}\mathbf{u} = \sum_j d\bar{t}_j \wedge \nu_j + \sum_j \eta_j \wedge dt_j$$

where η_j are of bidegree $(n-1, 1)$ and ν_j are of bidegree $(n, 0)$ whose restrictions to fibers are holomorphic (thus ν_j define sections of the bundle V). The Hermitian holomorphic vector bundle (V, H) admits the Chern connection $D = D' + D''$. The $(0,1)$-part D'' is given by $D''u = \sum_j \nu_j d\bar{t}_j$. Therefore, a section u of V is holomorphic if and only if $\bar{\partial}\mathbf{u} = \sum \eta_j \wedge dt_j$.

For the $(1,0)$-part D', we consider some local frame e of L and write $\mathbf{u} = u' \otimes e$ with u' an $(n, 0)$-form. Denote $h(e, e) = e^{-\phi}$ and define

$$\partial^{\phi}\mathbf{u} := (\partial^{\phi}u') \otimes e = e^{\phi}\partial(e^{-\phi}u') \otimes e \tag{5}$$

which is an $(n+1, 0)$-form with values in L. It is straightforward to check that (5) is independent of the choice of the frame e. We can write $\partial^{\phi}\mathbf{u} = \sum_j dt_j \wedge \mu_j$ where μ_j are of bidegree $(n, 0)$. If we denote by $P(\mu_j)$ the orthogonal projection of μ_j on the space of holomorphic forms on each fiber, then $D'u = \sum_j P(\mu_j)dt_j$.

Next, we are going to prove (V, H) is uniformly RC-positive. Fix a point $t_0 \in Y$. By the assumption in Theorem 4, there exists a nonzero tangent vector $v_0 \in T_{t_0}Y$ such that for any lift $\tilde{v}$ of v_0 to $T_{(t_0,z)}\mathfrak{X}$, we have $\Theta(\tilde{v}, \bar{\tilde{v}}) > 0$. So, the goal is to show that the curvature Θ^V of (V, H) satisfies $H(\Theta^V u_0, u_0)(v_0, \bar{v}_0) > 0$ for any nonzero vector u_0 in V_{t_0}.

We choose a coordinate system $(t_1, \ldots, t_m)$ around the point t_0 in Y such that $v_0 = \partial/\partial t_1$ at t_0. Consider a fixed $u_0 \neq 0$ in V_{t_0}. A standard argument allows us to extend u_0 to a local holomorphic section u of V such that $D'u = 0$ at t_0 and $u(t_0) = u_0 \neq 0$. A straightforward computation gives

$$\partial\bar{\partial}H(u, u) = -H(\Theta^V u, u) \text{ at } t_0. \tag{6}$$

On the other hand, if we let $\mathbf{u}$ be a representative of u and write $\mathbf{u} = u' \otimes e$ with u' an $(n, 0)$-form and e some local frame of L, then

$$H(u, u) = p_*(c_n u' \wedge \overline{u'} e^{-\phi}) \tag{7}$$

where $e^{-\phi} = h(e, e)$ and $c_n = i^{n^2}$. According to [Ber09, Proposition 4.2], we can choose a representative $\mathbf{u}$ such that in $\bar{\partial}\mathbf{u} = \sum \eta_j \wedge dt_j$, the η_j is primitive on $\mathfrak{X}_{t_0}$. Moreover, $\partial^{\phi}u' = 0$ at t_0. After using such a representative, we obtain

$$\partial\bar{\partial}H(u, u) = -c_n p_*(u' \wedge \overline{u'} \wedge \partial\bar{\partial}\phi e^{-\phi}) + (-1)^n c_n p_*(\bar{\partial}u' \wedge \overline{\bar{\partial}u'} e^{-\phi}) \text{ at } t_0. \tag{8}$$

We apply the above $(1,1)$-form to the tangent vector $v_0 = \partial/\partial t_1$ and get

$$\partial\bar{\partial}H(u, u)(v_0, \bar{v}_0) = -c_n p_*(u' \wedge \overline{u'} \wedge \partial\bar{\partial}\phi e^{-\phi})(v_0, \bar{v}_0) + (-1)^n c_n p_*(\bar{\partial}u' \wedge \overline{\bar{\partial}u'} e^{-\phi})(v_0, \bar{v}_0) \text{ at } t_0. \tag{9}$$

Because $\bar{\partial}\mathbf{u} = \sum \eta_j \wedge dt_j$ and $\mathbf{u} = u' \otimes e$, we see $\sum \eta_j \wedge dt_j = \bar{\partial}\mathbf{u} = \bar{\partial}u' \otimes e$. If we write $\eta_j = \eta'_j \otimes e$, then $\bar{\partial}u' = \sum \eta'_j \wedge dt_j$. So the last term in (9) is equal to

$$(-1)^n c_n \int_{\mathcal{X}_{t_0}} (-1)^n \sum \eta'_j \wedge \overline{\eta'}_k \wedge dt_j \wedge d\bar{t}_k e^{-\phi}(v_0, \bar{v}_0) = c_n \int_{\mathcal{X}_{t_0}} \eta'_1 \wedge \overline{\eta'}_1 e^{-\phi} \leq 0; \tag{10}$$

the last inequality is by the fact that the η_1 is primitive on $\mathcal{X}_{t_0}$.

We claim that the middle term in (9) is negative. For the $(n,0)$-form u', we can write locally

$$u' = u_z dz + \sum_{\lambda,j} u_{z_\lambda t_j} d\hat{z}_\lambda \wedge dt_j + (\text{terms with more than one } t_j), \tag{11}$$

where $dz = dz_1 \wedge \cdots \wedge dz_n$ and $d\hat{z}_\lambda$ means $dz_1 \wedge \cdots \wedge dz_n$ omitting dz_λ; here u_z and $u_{z_\lambda t_j}$ simply mean the coefficients, not differentiation. By a degree count, we see that $-c_n p_*(u' \wedge \overline{u'} \wedge \partial\bar{\partial}\phi e^{-\phi})$ is equal to

$$\begin{aligned}
-c_n \int_{\mathcal{X}_{t_0}} e^{-\phi} \Big(& \sum_{j,k} u_z dz \wedge \overline{u_z dz} \wedge \phi_{j\bar{k}} dt_j \wedge d\bar{t}_k \\
& + \sum_{\lambda,j,k} u_z dz \wedge \overline{u_{z_\lambda t_k} d\hat{z}_\lambda \wedge dt_k} \wedge \phi_{j\bar{\lambda}} dt_j \wedge d\bar{z}_\lambda \\
& + \sum_{\lambda,j,k} u_{z_\lambda t_j} d\hat{z}_\lambda \wedge dt_j \wedge \overline{u_z dz} \wedge \phi_{\lambda\bar{k}} dz_\lambda \wedge d\bar{t}_k \\
& + \sum_{\lambda,\mu,j,k} u_{z_\lambda t_j} d\hat{z}_\lambda \wedge dt_j \wedge \overline{u_{z_\mu t_k} d\hat{z}_\mu \wedge dt_k} \wedge \phi_{\lambda\bar{\mu}} dz_\lambda \wedge d\bar{z}_\mu \Big)
\end{aligned}$$

which can be organized as

$$\begin{aligned}
-c_n \sum_{j,k} \int_{\mathcal{X}_{t_0}} e^{-\phi} \Big(& |u_z|^2 \phi_{j\bar{k}} + \sum_\lambda (-1)^{n-\lambda+1} u_z \overline{u_{z_\lambda t_k}} \phi_{j\bar{\lambda}} + \sum_\lambda (-1)^{n-\lambda+1} \overline{u_z} u_{z_\lambda t_j} \phi_{\lambda\bar{k}} \\
& + \sum_{\lambda,\mu} (-1)^{\lambda+\mu} u_{z_\lambda t_j} \overline{u_{z_\mu t_k}} \phi_{\lambda\bar{\mu}} \Big) dz \wedge d\bar{z} \wedge dt_j \wedge d\bar{t}_k.
\end{aligned} \tag{12}$$

Applying the above $(1,1)$-form to $(v_0, \bar{v}_0)$, we see that $-c_n p_*(u' \wedge \overline{u'} \wedge \partial\bar{\partial}\phi e^{-\phi})(v_0, \bar{v}_0)$ is equal to

$$\begin{aligned}
-c_n \int_{\mathcal{X}_{t_0}} e^{-\phi} \Big(& |u_z|^2 \phi_{1\bar{1}} + \sum_\lambda (-1)^{n-\lambda+1} u_z \overline{u_{z_\lambda t_1}} \phi_{1\bar{\lambda}} + \sum_\lambda (-1)^{n-\lambda+1} \overline{u_z} u_{z_\lambda t_1} \phi_{\lambda\bar{1}} \\
& + \sum_{\lambda,\mu} (-1)^{\lambda+\mu} u_{z_\lambda t_1} \overline{u_{z_\mu t_1}} \phi_{\lambda\bar{\mu}} \Big) dz \wedge d\bar{z}.
\end{aligned} \tag{13}$$

If we denote the matrix $(\phi_{\lambda\bar{\mu}})$ by A, the column vector $(\phi_{\lambda\bar{1}})$ by ϕ_1, and the column vector $((-1)^{n-\lambda+1} u_{z_\lambda t_1})$ by B_1, then the expression inside the big parenthesis in (13) can be written as

$$|u_z|^2 \phi_{1\bar{1}} - \sum_{\lambda,\mu} \phi_{\lambda\bar{1}} \phi^{\lambda\bar{\mu}} \phi_{1\bar{\mu}} |u_z|^2 + \|\sqrt{A^{-1}} \phi_1 \overline{u_z} + \sqrt{A} B_1\|^2. \tag{14}$$

Combining (13) and (14), we get

$$
\begin{aligned}
&- c_n p_*(u' \wedge \overline{u'} \wedge \partial\bar{\partial}\phi e^{-\phi})(v_0, \bar{v}_0) \\
(15) \qquad &= - c_n \int_{\mathcal{X}_{t_0}} e^{-\phi}\Big(|u_z|^2(\phi_{1\bar{1}} - \sum_{\lambda,\mu} \phi_{\lambda\bar{1}}\phi^{\lambda\bar{\mu}}\phi_{1\bar{\mu}}) + \|\sqrt{A^{-1}}\phi_1\overline{u_z} + \sqrt{A}\overline{B_1}\|^2\Big) dz \wedge d\bar{z}.
\end{aligned}
$$

Meanwhile, by Lemma 5, the assumption in Theorem 4 implies that $\phi_{1\bar{1}} - \sum_{\lambda,\mu} \phi_{\lambda\bar{1}}\phi^{\lambda\bar{\mu}}\phi_{1\bar{\mu}} > 0$ on the fiber $\mathcal{X}_{t_0}$, so the integrand in (15) is semipositive on the fiber $\mathcal{X}_{t_0}$. Actually, the integrand in (15) is positive somewhere on the fiber $\mathcal{X}_{t_0}$. This is because $u(t_0) \neq 0$, $\mathbf{u}|_{\mathcal{X}_{t_0}}$ cannot be identically zero, hence u_z is nonzero somewhere.

As a consequence, from formula (15), we deduce that $-c_n p_*(u' \wedge \overline{u'} \wedge \partial\bar{\partial}\phi e^{-\phi})(v_0, \bar{v}_0)$ is negative, as claimed. All in all, the right hand side in (9) is negative, so $\partial\bar{\partial}H(u,u)(v_0,\bar{v}_0) < 0$ at t_0. By (6), we get $H(\Theta^V u_0, u_0)(v_0, \bar{v}_0) > 0$.

4 Weak RC-positivity

We first give a characterization of weak RC-positivity. It is a variant of [AV65, Lemma 18], [Yan19, Proposition 2.2], and [Dem12, Page 337]. Let $p : \mathcal{X}^{m+n} \to Y^m$ be the fibration with the Hermitian line bundle $(L, h) \to \mathcal{X}$ in the introduction. We consider $\beta \in C^\infty(\mathcal{X}, p^*(\wedge^{1,1}T^*Y))$ and call β positive if the matrix $(\beta_{i\bar{j}})$ in the local expression $i\sum_{i,j} \beta_{i\bar{j}} dt_i \wedge d\bar{t}_j$ is positive.

Lemma 6. *Assume the curvature Θ of h is positive on every fiber $\mathcal{X}_t$. For $1 \leq k \leq m$, the following are equivalent.*

A. The curvature Θ has at least $n+k$ positive eigenvalues at every point in $\mathcal{X}$.

B. The horizontal component $\Theta_{\mathcal{H}}$ has at least k positive eigenvalues at every point in $\mathcal{X}$.

C. There exists a positive $\beta \in C^\infty(\mathcal{X}, p^*(\wedge^{1,1}T^*Y))$ such that the sum of any $m-k+1$ eigenvalues of $\Theta_{\mathcal{H}}$ with respect to β is positive.

Moreover, when $k = 1$, statement C can be rephrased as $\Theta_{\mathcal{H}} \wedge \beta^{m-1} > 0$ or equivalently $\Theta^{n+1} \wedge \beta^{m-1} > 0$.

Proof. The equivalence between A and B can be seen by the decomposition $\Theta = \Theta_{\mathcal{H}} + \Theta_{\mathcal{V}}$ and formulas (2) and (3). To prove that C implies B, we denote the eigenvalues of $\Theta_{\mathcal{H}}$ with respect to β by $\gamma_1 \leq \ldots \leq \gamma_m$. Since $0 < \gamma_1 + \cdots + \gamma_{m-k+1}$, we see that γ_{m-k+1} must be positive, so $\Theta_{\mathcal{H}}$ has at least k positive eigenvalues.

To prove that B implies C, we first fix a positive β_0 in $C^\infty(\mathcal{X}, p^*(\wedge^{1,1}T^*Y))$ and denote the eigenvalues of $\Theta_{\mathcal{H}}$ with respect to β_0 by $\gamma_1 \leq \ldots \leq \gamma_m$. We know that $\gamma_{m-k+1} > 0$ since there are at least k positive eigenvalues for $\Theta_{\mathcal{H}}$. We are going to construct β using the arguments in [Dem12, Page 337] (see also [AV65, Lemma 18] and [Yan19, Proposition 2.2]).

Consider the positive numbers $A := \inf_{\mathcal{X}} \gamma_{m-k+1} > 0$, $B := \sup_{\mathcal{X}} \max_j |\gamma_j| > 0$, and $\varepsilon := 1/(m-k+1)$. Let ψ_ε be in $C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$\psi_\varepsilon(x) = x \text{ for } x \geq A, \;\; \psi_\varepsilon(x) \geq x \text{ for } 0 \leq x \leq A, \;\; \psi_\varepsilon(t) = B/\varepsilon \text{ for } x \leq 0. \tag{16}$$

By [Dem12, Page 337, Lemma 5.2], $\psi_\varepsilon(\Theta_{\mathcal{H}})$ is in $C^\infty(\mathcal{X}, p^*(\wedge^{1,1}T^*Y))$ and positive. For a point $(t,z) \in \mathcal{X}$, if $\{\xi_1, \ldots, \xi_m\}$ is a basis of T_t^*Y such that $\beta_0(t,z) = \sum \xi_j \wedge \bar{\xi}_j$, then

$$\Theta_{\mathcal{H}}(t,z) = \sum \gamma_j \xi_j \wedge \bar{\xi}_j \text{ and } \psi_\varepsilon(\Theta_{\mathcal{H}})(t,z) = \sum \psi_\varepsilon(\gamma_j)\xi_j \wedge \bar{\xi}_j. \tag{17}$$

Therefore, the eigenvalues of $\Theta_{\mathcal{H}}$ with respect to $\psi_\varepsilon(\Theta_{\mathcal{H}})$ are $\gamma_j/\psi_\varepsilon(\gamma_j)$. For $1 \leq j \leq m$,

$$\begin{cases} \text{if } 0 \leq \gamma_j, \text{ then } \gamma_j \leq \psi_\varepsilon(\gamma_j) \text{ and } 0 \leq \gamma_j/\psi_\varepsilon(\gamma_j) \leq 1. \\ \text{if } \gamma_j < 0, \text{ then } \psi_\varepsilon(\gamma_j) = B/\varepsilon \text{ and } -\varepsilon \leq \gamma_j/(B/\varepsilon) \leq 0. \end{cases}$$

All in all, $-\varepsilon \leq \gamma_j/\psi_\varepsilon(\gamma_j) \leq 1$ for $1 \leq j \leq m$. On the other hand, for $m-k+1 \leq j \leq m$, we have $A \leq \gamma_j$, hence $\psi_\varepsilon(\gamma_j) = \gamma_j$ and $\gamma_j/\psi_\varepsilon(\gamma_j) = 1$.

As a result, the sum of any $m-k+1$ eigenvalues of $\Theta_{\mathcal{H}}$ with resect to $\psi_\varepsilon(\Theta_{\mathcal{H}})$ is at least $1 - \varepsilon(m-k) = \varepsilon > 0$ as we chose $\varepsilon = 1/(m-k+1)$. By setting $\beta = \psi_\varepsilon(\Theta_{\mathcal{H}})$, the proof is complete.

For the moreover part, it is clear that, when $k=1$, statement C can be rephrased as $\Theta_{\mathcal{H}} \wedge \beta^{m-1} > 0$. For the equivalence to $\Theta^{n+1} \wedge \beta^{m-1} > 0$, we will use the following formula:

$$\Theta^{n+1} \wedge \beta^{m-1} = (n+1)!(m-1)! \sum_{i,j} \beta^{i\bar{j}} \det M(i\bar{j}) \det(\beta) \Big(\bigwedge_{k=1}^{m} idt_k \wedge d\bar{t}_k \wedge \bigwedge_{\lambda=1}^{n} idz_\lambda \wedge d\bar{z}_\lambda \Big). \tag{18}$$

Here $M(i\bar{j})$ for fixed i and j is a matrix defined by

$$M(i\bar{j}) := \begin{pmatrix} \phi_{t_i\bar{t}_j} & \phi_{t_i\bar{z}_1} & \cdots & \phi_{t_i\bar{z}_n} \\ \phi_{z_1\bar{t}_j} & \phi_{z_1\bar{z}_1} & \cdots & \phi_{z_1\bar{z}_n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{z_n\bar{t}_j} & \phi_{z_n\bar{z}_1} & \cdots & \phi_{z_n\bar{z}_n} \end{pmatrix} \tag{19}$$

with respect to local coordinates $(t_1, \ldots, t_m)$ in Y and local coordinates $(t_1, \ldots, t_m, z_1, \ldots, z_n)$ in $\mathcal{X}$, and ϕ is a local weight for the metric h, namely $h = e^{-\phi}$. Formula (18) is a variant of formula (3) in [Wu25]. To prove formula (18), it suffices to consider a fixed point (t,z) in $\mathcal{X}$. Let α be a Hermitian metric on Y such that $p^*\alpha = \beta$ at (t,z). We then apply formula (3) in [Wu25] to deduce formula (18). (Formula (18) is related to the Wess–Zumino–Witten equation, see [Wu23c, Wu26a, Fin24a, Fin24b]).

According to Schur's formula, we have

$$\phi_{i\bar{j}} - \sum_{\lambda,\mu} \phi_{i\bar{\mu}} \phi^{\lambda\bar{\mu}} \phi_{\lambda\bar{j}} = \frac{\det M(i\bar{j})}{\det(\phi_{\lambda\bar{\mu}})}. \tag{20}$$

A reminder on the notation: $M(i\bar{j})$ for fixed i and j is itself a matrix given in (19), but $(\phi_{\lambda\bar{\mu}})$ is a matrix with (λ, μ) entry equal to $\phi_{\lambda\bar{\mu}}$. By formula (18), the sign of $\Theta^{n+1} \wedge \beta^{m-1}$ is decided by $\sum_{i,j} \beta^{i\bar{j}} \det M(i\bar{j})$ which has the same sign with

$$\sum_{i,j} \beta^{i\bar{j}}(\phi_{i\bar{j}} - \sum_{\lambda,\mu} \phi_{i\bar{\mu}}\phi^{\lambda\bar{\mu}}\phi_{\lambda\bar{j}}) \tag{21}$$

by formula (20) and the fact $(\phi_{\lambda\bar{\mu}})$ is positive definite. But the sign of $\Theta_{\mathcal{H}} \wedge \beta^{m-1}$ is also decided by (21) according to (2). All together, $\Theta^{n+1} \wedge \beta^{m-1} > 0$ if and only if $\Theta_{\mathcal{H}} \wedge \beta^{m-1} > 0$. ■

The case we care about most in this paper is when $k = 1$ in Lemma 6 because it corresponds to weak RC-positivity. The difficulty in proving a theorem like Theorem 3 or Theorem 4 for weak RC-positivity is that the β in Lemma 6 is on $\mathcal{X}$, so it does not quite fit into Berndtsson's computation, especially formula (15). So, we raise the question:

Question. *Is it possible to choose β in Lemma 6 so that $\beta = p^*\alpha$ for some Hermitian metric α on Y?*

This question is somewhat bold because if it is possible to choose $\beta = p^*\alpha$, then we can use [Wu25, Theorem 4] to deduce that if E is weakly RC-positive, then $S^k \otimes \det E$ has positive mean curvature for $k \geq 0$. Moreover, it is even possible to use [Wu25, Theorem 5] to deduce that if E is weakly RC-positive, then E has positive mean curvature. Since positive mean curvature implies RC-positivity ([Yan18, Theorem 3.6]), this would mean that RC-positivity, weak RC-positivity, and mean curvature positivity are all equivalent. Such an equivalence is conjectured for tangent bundle TX in [Yan20, Problems 4.15 and 4.17].

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