ON IRREDUCIBILITY FOR A CLASS OF PARAMETERIZED MULTIPLICATION OPERATORS BY POLYNOMIALS

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Abstract: We study the reducibility of multiplication operators defined by quasi-homogeneous polynomials on the weighted Hardy space $H^2(\omega, \delta)$. The study of these operators essentially reduces to that of the parameterized multiplication operators $M_{z+\alpha w}$ for $\alpha \in (0, +\infty)$. By employing determinant techniques and algebraic methods, we prove that the operator $M_{z+\alpha w}$ is reducible for at most a countable subset of α. Furthermore, we construct examples to show that such a set of α can be empty or a singleton. Our results also provide support for a conjecture due to Guo and Wang on the reducibility of M_{z+w}.

Keywords: Weighted Hardy spaces; Multiplication operators; Reducibility.

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1. Introduction

The study of reducing subspaces of multiplication operators on function spaces represents one of the central topics in operator theory, closely related to the structure of von Neumann algebras. Since the 1970s, this field has undergone two significant developmental phases. The first involved early research focused on the commutants and reducing subspaces of multiplication operators on the Hardy space $H^2(\mathbb{D})$ over the unit disk (see [1, 2, 3, 27, 26]), while the second phase began with Zhu's conjecture regarding the minimal reducing subspaces of multiplication operators induced by finite Blaschke products. This breakthrough significantly advanced research on related problems in Bergman spaces and other function spaces. Along this research trajectory, substantial results have emerged over the past four decades (see [4, 6, 7, 9, 10, 11, 12, 13, 15, 18, 24, 25, 29, 30, 31]), and a comprehensive overview can be found in [14]. The characterization of reducing subspaces for multiplication operators induced by general multivariate symbols has emerged as a research focus. We draw the readers' attention to [19, 21, 20, 5].

Among various multiplication operators, those induced by the polynomial $p(z, w) = z^k + \alpha w^l$ ($\alpha \neq 0$) are of special interest. Wang, Dan, and Huang [28] investigated the reducing subspace structure of multiplication operators defined by polynomials of the form $\alpha z^k + \beta w^l$. This essentially reduces to the situation of $z^k + \alpha w^l$ ($\alpha > 0$), where it is shown that if $p_\alpha = z^k + \alpha w^l$ with $0 < \alpha < 1$, then the von Neumann algebra $\mathcal{V}^*(p)$ ($= \{M_p, M_p^*\}'$) is abelian and of dimension kl [28]. For $\alpha = 1$, the situation is completely characterized in [5]. Subsequently, Guo and Wang [17] systematically established the theory of graded structures induced by operators on Hilbert spaces. By applying $\mathcal{S}$-calculus (see Section 2) to any bounded operator T, they generate a family of operators $\{\mathcal{S}^n\}_{n\in\mathbb{Z}}$. This allowed them to decompose the Hilbert space H into a homogeneous structure $H = \bigoplus_n H_n$ such that $TH_n \subseteq$

H_{n+1}. Using graded module theory, they made new discoveries regarding reducing subspaces for generic operators.

We follow the notation in [16]. Let $\omega = \{\omega_0, \omega_1, \cdots, \omega_n, \cdots\}$ be a sequence of positive numbers. The weighted Hardy space $H^2(\omega)$ is the Hilbert space of analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ equipped with the norm

$$\|f\|_\omega^2 = \sum_{k=0}^{\infty} \omega_k |a_k|^2 < \infty.$$

Setting $\omega_n = \|z^n\|^2$, the set $\{\frac{z^n}{\sqrt{\omega_n}} : n \geq 0\}$ forms an orthonormal basis for $H^2(\omega)$. Each function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in $H^2(\omega)$ is analytic on $\{z \in \mathbb{C} : |z| < R\}$, where $R = \liminf_{n\to\infty} (\omega_n)^{1/n}$ (see, for example, [23, Theorem 10]).

For a sequence $\delta = \{\delta_n\}$ of positive numbers, we can define $H^2(\delta)$ in an analogous way. We identify the tensor product space $H^2(\omega) \otimes H^2(\delta)$ with the Hilbert space of analytic functions in two variables $f(z, w) = \sum_{k,l=0}^{\infty} a_{kl} z^k w^l$ equipped with the norm

$$\|f\|_{\omega,\delta}^2 = \sum_{k,l=0}^{\infty} \omega_k \delta_l |a_{kl}|^2 < \infty.$$

For brevity, we denote $H^2(\omega) \otimes H^2(\delta)$ by $H^2(\omega, \delta)$. Our main focus is on the reducibility of multiplication operators defined by quasi-homogeneous polynomials on the weighted Hardy space $H^2(\omega, \delta)$. Recall that for positive integers m, n with $\gcd(m, n) = 1$, a polynomial $p(z, w) = \sum a_{i,j} z^i w^j$ is called *quasi-homogeneous with the weight* (m, n) if there is a positive integer L such that $mi + nj = L$ for all (i, j) with $a_{i,j} \neq 0$. This integer L is called the degree of p with respect to the weight (m, n). The polynomials of the form $z^k + \alpha w^l$ are quasi-homogeneous.

The following example shows that the study of multiplication operators defined by quasi-homogeneous polynomials on $H^2(\omega, \delta)$ is reduced to a specialized class of multiplication operators.

Example 1.1. *Let m and n be positive integers with $\gcd(m, n) = 1$, and let q be a quasi-homogeneous polynomial in $\mathbb{C}[z, w]$ with weight (m, n) and degree L. We will show that if $L \geq 2$, then M_q is reducible on $H^2(\omega, \delta)$.*

To see this, let M be the closed span of all polynomials p of degree kL for $k = 0, 1, 2, \cdots$. It is routine to verify that M is invariant under both M_q and M_q^*, since M_q acting on a polynomial of degree kL increases its degree by L, while M_q^* decreases its degree by L. Since $L = mi + nj$ for some nonnegative integers i, j with $\gcd(m, n) = 1$, and $L \geq 2$, it follows that at least one of m and n is not a multiple of L. Otherwise, we would have $\gcd(m, n) \geq L$, which is a contradiction. Therefore, either $z \notin M$ or $w \notin M$. This implies that M is a nontrivial reducing subspace of M_q, ensuring the reducibility of M_q.

If $L = 1$, it is immediate that q must be a linear combination of z and w. Therefore, it suffices to focus on the case where $q(z, w) = z + \alpha w$ for some $\alpha \in \mathbb{C}\backslash\{0\}$. Without loss of generality, we may restrict our attention to $\alpha \in (0, +\infty)$.

Note that for $\alpha \in (0, +\infty)$, $M_{z+\alpha w}$ is reducible on $L_a^2(\mathbb{D}^2)$ if and only if $\alpha = 1$ (see [5, 28]). It is reasonable to raise the following question:

For α in $(0, +\infty)$, is it true that $M_{z+\alpha w}$ is reducible on $H^2(\omega, \delta)$ for at most a Lebesgue null set of α ?

Throughout this paper, we assume that the operators M_z and M_w are bounded. Specifically, this requires the following conditions on the weight sequences:

$$\sup_n \frac{\omega_{n+1}}{\omega_n} < \infty \text{ and } \sup_m \frac{\delta_{m+1}}{\delta_m} < \infty. \tag{1.1}$$

The following is our main result, which provides an affirmative answer to the above question.

Theorem 1.2. *Under the assumption (1.1), for $\alpha \in (0, +\infty)$, the multiplication operator $M_{z+\alpha w}$ is reducible on the weighted Hardy space $H^2(\omega, \delta)$ for at most a countable subset of α, which may be empty.*

Let $\mathcal{Q}_\alpha$ denote the set of all non-commutative polynomials in $M_{z+\alpha w}$ and $M^*_{z+\alpha w}$. If $M_{z+\alpha w}$ is irreducible, then $\{M_{z+\alpha w}, M^*_{z+\alpha w}\}' = \mathbb{C}I$. By the von Neumann Bicommutant Theorem, the strong operator topology (SOT) closure of $\mathcal{Q}_\alpha$ satisfies:

$$\overline{\mathcal{Q}_\alpha}^{\text{SOT}} = \{M_{z+\alpha w}, M^*_{z+\alpha w}\}'' = (\mathbb{C}I)' = B(H^2(\omega, \delta)).$$

Therefore, under the assumption (1.1), Theorem 1.2 implies that for all but countably many $\alpha \in (0, +\infty)$, each operator in $B(H^2(\omega, \delta))$ can be approximated in SOT by a net of non-commutative polynomials in $M_{z+\alpha w}$ and $M^*_{z+\alpha w}$ from $\mathcal{Q}_\alpha$.

For a weighted sequence $\{a_n\}_{n=0}^\infty$ with $a_n > 0$, we define the unilateral weighted shift operator A by $Ae_n = a_n e_{n+1}$ for $n \geq 0$, where $\{e_n : n \geq 0\}$ is an orthonormal basis of a Hilbert space H. Let B be another unilateral weighted shift operator on a Hilbert space K. Then, for any $\alpha \in \mathbb{C}$, the operator $A \otimes I + \alpha I \otimes B$ is unitarily equivalent to $M_{z+\alpha w}$ on $H^2(\omega, \delta)$. With this notation, Theorem 1.2 can then be restated as follows: for $\alpha \in (0, +\infty)$, the operator $A \otimes I + \alpha I \otimes B$ on $H \otimes K$ is reducible for at most a countable subset of α.

This countable set in Theorem 1.2 can be either empty or a singleton (see Subsections 5.1 and 5.2). However, we have not yet identified such a set containing two distinct points in $(0, +\infty)$. The difficulty in this investigation is partially attributed to Conjecture 4.8 in [17], stated as below.

Conjecture. *The operator M_{z+w} on $H^2(\omega, \delta)$ is reducible only if $\omega \sim \delta$; that is, $\frac{\omega_{n+1}}{\omega_n} = \frac{\delta_{n+1}}{\delta_n}$ for $n \geq 0$.*

Remark 1.3. *Observe that for $\alpha_1, \alpha_2 \in (0, +\infty)$, the operators $\alpha_1 M_w$ and $\alpha_2 M_w$ are not unitarily equivalent, and that αM_w can be represented as M_w on some Hilbert space $H^2(\widetilde{\delta}(\alpha))$, where $\widetilde{\delta}(\alpha)$ is a weight depending on α. Therefore, M_z is unitarily equivalent to αM_w for at most one $\alpha \in (0, +\infty)$. Consequently, if this conjecture holds, then on the space $H^2(\omega, \delta)$ the operator $M_{z+\alpha w}$ is reducible for at most one point $\alpha \in (0, +\infty)$. Conversely, if there existed a Hilbert space $H^2(\omega, \delta)$ on which $M_{z+\alpha w}$ is reducible for two distinct points $\alpha \in (0, +\infty)$, then this conjecture would fail.*

Remark 1.4. *A unilateral weighted shift A is called simple if its weights satisfy certain conditions. Specifically, for the multiplication operator M_w on $H^2(\delta)$ this condition exactly requires $P(n+1) - P(n) \neq 0$ for $n \geq 1$ (where $P(n)$ is defined in Section 2).*

Guo and Wang proved that the aforementioned conjecture holds for simple unilateral weighted shifts [16].

The proof of Theorem 1.2 relies on the following result, established through determinant-based techniques and algebraic methods.

Theorem 1.5. *Suppose there exists $k \geq 2$ such that*

$$P(1) = P(2) = \cdots = P(k-1) = 0 \quad \textit{and} \quad P(k) \neq 0.$$

Then for $\alpha \in (0, +\infty)$, $M_{z+\alpha w}$ is reducible on the weighted Hardy space $H^2(\omega, \delta)$ for at most a countable subset $\mathcal{A}_k$ of α.

The paper is organized as follows. Section 2 presents some preliminaries. Section 3 proves special cases of Theorem 1.5 as illustrative examples. Section 4 contains the proofs of Theorems 1.5 and 1.2. Finally, Section 5 demonstrates through examples that the countable set in Theorem 1.2 can be empty or a singleton.

2. Preliminaries

First, we introduce some notations. For a fixed $\alpha > 0$, set $p_\alpha = z + \alpha w$, and $M_{p_\alpha} = M_z + \alpha M_w$. We define

$$T = [M_{p_\alpha}^*, M_{p_\alpha}] = M_{p_\alpha}^* M_{p_\alpha} - M_{p_\alpha} M_{p_\alpha}^*.$$

By computation, we have

$$T = (M_z^* M_z - M_z M_z^*) + \alpha^2 (M_w^* M_w - M_w M_w^*).$$

For $m \geq 0$, we define

$$\varphi(m) = \frac{\omega_{m+1}}{\omega_m}, \quad \psi(m) = \frac{\delta_{m+1}}{\delta_m},$$

and set

$$\varphi(-1) = \psi(-1) = \varphi(-2) = \psi(-2) = \cdots = 0.$$

For all $n \in \mathbb{Z}$, we define

$$\nabla\varphi(n) = \varphi(n) - \varphi(n-1), \quad \nabla\psi(n) = \psi(n) - \psi(n-1).$$

Note that

$$\nabla\varphi(0) = \varphi(0), \quad \nabla\psi(0) = \psi(0).$$

Furthermore, for $n \geq 1$, we define

$$\nabla^2\psi(n) = \nabla\psi(n) - \nabla\psi(n-1).$$

Rewrite $P(n) = \nabla^2\psi(n)$.

Lemma 2.1. *With the above notation, for all $m, n \geq 0$,*

$$T z^m w^n = \lambda_\alpha(m,n) z^m w^n,$$

where

$$\lambda_\alpha(m,n) = \nabla\varphi(m) + \alpha^2 \nabla\psi(n).$$

Proof. For $m \geq 1$, there exists a constant c_m such that

$$M_z^* z^m = c_m z^{m-1}.$$

To determine the constant c_m, we proceed as follows

$$\|z^m\|^2 = \langle M_z z^{m-1}, z^m\rangle = \langle z^{m-1}, M_z^* z^m\rangle = \langle z^{m-1}, c_m z^{m-1}\rangle = c_m \|z^{m-1}\|^2,$$

which yields

$$c_m = \frac{\|z^m\|^2}{\|z^{m-1}\|^2} = \frac{\omega_m}{\omega_{m-1}}.$$

Similarly, for $n \geq 1$ we have

$$M_w^* w^n = \frac{\delta_n}{\delta_{n-1}} w^{n-1}.$$

Now, for $m, n \geq 0$, we have

$$\begin{aligned}[M_z^*, M_z] z^m w^n &= M_z^* M_z z^m w^n - M_z M_z^* z^m w^n \\ &= \left(\frac{\omega_{m+1}}{\omega_m} - \frac{\omega_m}{\omega_{m-1}} \right) z^m w^n.\end{aligned}$$

Similarly,

$$[M_w^*, M_w] z^m w^n = \left(\frac{\delta_{n+1}}{\delta_n} - \frac{\delta_n}{\delta_{n-1}} \right) z^m w^n.$$

Therefore,

$$\begin{aligned}T z^m w^n &= [M_z^*, M_z] z^m w^n + \alpha^2 [M_w^*, M_w] z^m w^n \\ &= \nabla\varphi(m) + \alpha^2 \nabla\psi(n) \\ &= \lambda_\alpha(m, n) z^m w^n,\end{aligned}$$

where we adopt the convention that $\frac{\omega_0}{\omega_{-1}} = \frac{\delta_0}{\delta_{-1}} = 0$. □

Let $\mathbb{Z}_+$ denote the set of non-negative integers. Following [5], for each subset F of $H^2(\omega, \delta)$, for fixed α define $\mathcal{S}F$ to be the closed span of

$$\left\{ \prod_{k=1}^{m} M_{p_\alpha}^{i_k} M_{p_\alpha}^{*j_k} x : i_k, j_k \in \mathbb{Z}_+, \sum_{k=1}^{m} (i_k - j_k) = 1, x \in F \right\}.$$

For each $r \in \mathbb{Z}_+$, we define

$$E_r = \operatorname{span}\{z^n w^m : n + m = r\}.$$

It is easy to show that $\dim E_r = r + 1$ and $H^2(\omega, \delta) = \bigoplus_{r=0}^{\infty} E_r$. Furthermore, $\mathcal{S}E_r \subseteq E_{r+1}$ for every $r \in \mathbb{Z}_+$. Following [5], we define an equivalence relation $\overset{\alpha}{\sim}$ on $\mathbb{Z}_+^2$ by

$$(m, n) \overset{\alpha}{\sim} (m', n') \text{ if and only if } \lambda_\alpha(m, n) = \lambda_\alpha(m', n').$$

The following lemma is from [28], and we include a proof for completeness.

Lemma 2.2. [28, Lemma 3.4] *Let $r \in \mathbb{Z}_+$. If*

$$(r + 1, 0) \overset{\alpha}{\sim} (r, 1) \overset{\alpha}{\sim} \cdots \overset{\alpha}{\sim} (1, r) \overset{\alpha}{\sim} (0, r + 1) \tag{2.1}$$

does not hold, then $\mathcal{S}E_r = E_{r+1}$.

Proof. The proof comes from [28].

Assume that (2.1) fails for some $r \in \mathbb{Z}_+$. Clearly, $\mathcal{S}E_r \subseteq E_{r+1}$. To establish the reverse inclusion, it suffices to show that

$$\dim \mathcal{S}E_r \geq r + 2 = \dim E_{r+1}$$

We first assume $r \geq 1$, and will prove that $TM_{z+\alpha w}(E_r) \nsubseteq M_{z+\alpha w}(E_r)$. Suppose to the contrary that $TM_{z+\alpha w}(E_r) \subseteq M_{z+\alpha w}(E_r)$ for $r \in \mathbb{Z}_+$. Then, for all $j = 0, 1, \ldots, r$, we have

$$T\{(z + \alpha w)(z^{r-j} w^j)\} \in M_{z+\alpha w}(E_r).$$

Recall that the action of T is given by

$$Tz^m w^n = \lambda_\alpha(m,n) z^m w^n \quad m,n \in \mathbb{Z}_+.$$

Thus, we compute

$$\begin{aligned} T\{(z+\alpha w)(z^{r-j}w^j)\} &= T(z^{r-j+1}w^j) + \alpha T(z^{r-j}w^{j+1}) \\ &= \lambda_\alpha(r-j+1,j) z^{r-j+1}w^j + \alpha\lambda_\alpha(r-j,j+1) z^{r-j}w^{j+1} \\ &= c_1(z+\alpha w)w^r + c_2(z+\alpha w)zw^{r-1} + \cdots + c_{r+1}(z+\alpha w)z^r \end{aligned}$$

When $j=r$, this becomes

$$\begin{aligned} T\{(z+\alpha w)(z^{r-j}w^j)\} &= \lambda_\alpha(1,r)zw^r + \alpha\lambda_\alpha(0,r+1)w^{r+1} \\ &= c_1(zw^r + \alpha w^{r+1}) + c_2(z^2w^{r-1} + \alpha zw^r) + \cdots + c_{r+1}(z^{r+1} + \alpha z^r w). \end{aligned}$$

Now suppose $\lambda_\alpha(1,r) \neq \lambda_\alpha(0,r+1)$. Then comparing coefficients forces $c_2 \neq 0$, $c_3 \neq 0, \cdots, c_{r+1} \neq 0$. This leads to a contradiction. Therefore, $\lambda_\alpha(1,r) = \lambda_\alpha(0,r+1)$; that is, $(r+1,0) \overset{\alpha}{\sim} (r,1)$. Similarly, we obtain

$$(r+1,0) \overset{\alpha}{\sim} (r,1) \overset{\alpha}{\sim} \cdots \overset{\alpha}{\sim} (1,r) \overset{\alpha}{\sim} (0,r+1)$$

to reach a contradiction. Hence, $TM_{z+\alpha w}E_r \nsubseteq M_{z+\alpha w}E_r$, and we conclude

$$\dim(\mathcal{S}E_r) \geq \dim(M_{z+\alpha w}(E_r)) + 1 = r+2, \quad r \geq 1.$$

Therefore, $\mathcal{S}E_r = E_{r+1}$.

Now consider the case $r=0$. By $(1,0) \overset{\alpha}{\nsim} (0,1)$, it is straightforward to verify that $M_{z+\alpha w}1$ and $TM_{z+\alpha w}$ are linearly independent in $\mathcal{S}E_0$. Then it follows that $\mathcal{S}E_0 = E_1$. The proof is complete. ☐

The following is from [28, Lemma 3.1].

Lemma 2.3. *If M is a reducing subspace for M_z satisfying* $\dim M \ominus \mathcal{S}M = 1$, *then M is a minimal reducing subspace. Consequently, if $\mathcal{S}E_r = E_{r+1}$ for $r \geq 0$, then M_{p_α} is irreducible.*

By induction, we can prove the following.

Lemma 2.4. *If $P(1) = P(2) = \cdots = P(k) = 0$, then $\psi(i) = (i+1)\psi(0)$ for all $1 \leq i \leq k$.*

Proposition 2.5. *Suppose M_w is bounded. Then $P(n)$ cannot be identically zero for all $n \geq 1$.*

Proof. Assume M_w is bounded. Then

$$\sup_{i \geq 0} \frac{\delta_{i+1}}{\delta_i} < \infty.$$

If $P(n) \equiv 0$ for all n, then in particular $P(1) = P(2) = \cdots = P(k) = 0$ for all k. Then Lemma 2.4 implies

$$\psi(i) = (i+1)\psi(0), \text{ for all } i \geq 1.$$

Recall that

$$\psi(i) = \frac{\delta_{i+1}}{\delta_i} \quad \text{and} \quad \psi(0) = \frac{\delta_1}{\delta_0}.$$

Substituting these into the above gives

$$\frac{\delta_{i+1}}{\delta_i} = (i+1)\frac{\delta_1}{\delta_0}, \text{ for all } i \geq 1.$$

But the right-hand side grows unbounded as $i \to \infty$, contradicting the boundedness of M_w. Hence $P(n)$ cannot be identically zero. □

3. Some special cases for Theorem 1.5

In this section, we discuss the cases $k = 1, 2$ and 3 for Theorem 1.5. Recall that $p_\alpha(z, w) = z + \alpha w, \alpha \in (0, +\infty)$.

Example 3.1. *Suppose $P(1) \neq 0$. Define the set*

$$\mathcal{A}_1 = \{\alpha \in (0, +\infty) \mid \alpha^2 = \frac{\nabla\varphi(r+1) - \nabla\varphi(r)}{\nabla\psi(1) - \psi(0)} \textit{ for some } r \in \mathbb{Z}_+\}.$$

Then for $\alpha \in (0, +\infty)$, M_{p_α} is reducible on $H^2(\omega, \delta)$ only if $\alpha \in \mathcal{A}_1$.

Proof. For each $\alpha \in (0, +\infty) \setminus \mathcal{A}_1$, we have

$$\nabla\varphi(r+1) - \nabla\varphi(r) \neq \alpha^2(\nabla\psi(1) - \psi(0)) \quad \text{for all } r \in \mathbb{Z}_+.$$

Rewriting this inequality yields

$$\nabla\varphi(r+1) + \alpha^2\psi(0) \neq \nabla\varphi(r) + \alpha^2\nabla\psi(1) \quad \text{for all } r \in \mathbb{Z}_+,$$

which implies $(r+1, 0) \overset{\alpha}{\nsim} (r, 1)$ for all $r \in \mathbb{Z}_+$.

By Lemma 2.2, it follows that $\mathcal{S}E_r = E_{r+1}$ for all $r \geq 0$. Then applying Lemma 2.3, we conclude that the multiplication operator M_{p_α} is reducible on $H^2(\omega, \delta)$ only if $\alpha \in \mathcal{A}_1$. □

Example 3.2. *Assume $P(1) = 0$ and $P(2) \neq 0$. Then there exists a countable set $\mathcal{A}_2$ such that for $\alpha \in (0, +\infty)$, the multiplication operator M_{p_α} is reducible on $H^2(\omega, \delta)$ only if $\alpha \in \mathcal{A}_2$.*

Proof. By Lemma 2.3, to prove that M_{p_α} is irreducible on $H^2(\omega, \delta)$ for $\alpha \in (0, +\infty) \setminus \mathcal{A}_2$, it suffices to show that $\mathcal{S}E_r = E_{r+1}$ for all $r \geq 0$ and such α.

Define the set

$$\mathcal{A}_{21} = \{\alpha \in (0, +\infty) \mid \alpha^2 = \frac{\nabla\varphi(r) - \nabla\varphi(r-1)}{\nabla\psi(2) - \nabla\psi(1)} \text{ for some } r \geq 1\}.$$

For $\alpha \notin \mathcal{A}_{21}$, we have for all $r \geq 1$ that

$$\nabla\varphi(r) - \nabla\varphi(r-1) \neq \alpha^2(\nabla\psi(2) - \nabla\psi(1)),$$

which implies

$$\nabla\varphi(r) + \alpha^2\nabla\psi(1) \neq \nabla\varphi(r-1) + \alpha^2\nabla\psi(2).$$

Therefore, $(r, 1) \overset{\alpha}{\nsim} (r-1, 2)$ for all $r \geq 1$. By Lemma 2.2, it follows that $\mathcal{S}E_r = E_{r+1}$ for all $r \geq 1$.

It remains to show that $\mathcal{S}E_0 = E_1$. Consider the functions

$$M_{p_\alpha}1 = z + \alpha w,$$

$$\begin{aligned} M_{p_\alpha}^{*2} M_{p_\alpha}^3 1 &= M_{z+\alpha w}^{*2}(z^3 + 3\alpha z^2 w + 3\alpha^2 z w^2 + \alpha^3 w^3) \\ &= D_1 z + D_2 w, \end{aligned}$$

where

$$\begin{aligned} D_1 &= \frac{\omega_3}{\omega_1} + \alpha^2 \frac{\omega_2}{\omega_1}\frac{\delta_1}{\delta_0} + 3\alpha^4 \frac{\delta_2}{\delta_0} \\ &= \varphi(2)\varphi(1) + 6\alpha^2\varphi(1)\psi(0) + 3\alpha^4\psi(1)\psi(0), \\ D_2 &= 3\frac{\omega_2}{\omega_0} + 6\alpha^2 \frac{\omega_1}{\omega_0}\frac{\delta_2}{\delta_1} + \alpha^5 \frac{\delta_3}{\delta_1} \\ &= 3\alpha\varphi(1)\varphi(0) + 6\alpha^3\varphi(0)\psi(1) + \alpha^5\psi(2)\psi(1). \end{aligned}$$

If $M_{p_\alpha}1$ and $M_{p_\alpha}^{*2}M_{p_\alpha}^3 1$ are linearly independent, then $\dim \mathcal{S}E_0 \geq 2$. Since $\mathcal{S}E_0 \subseteq E_1$ and $\dim E_1 = 2$, it follows that $\mathcal{S}E_0 = E_1$.

Now, to reach a contradiction, we assume they are linearly dependent. Then the determinant

$$J = \begin{vmatrix} 1 & \alpha \\ D_1 & D_2 \end{vmatrix} = 0.$$

Computing this determinant gives

$$\begin{aligned} J &= D_2 - \alpha D_1 \\ &= [\psi(2)\psi(1) - 3\psi(1)\psi(0)]\alpha^5 + [6\varphi(0)\psi(1) - 6\varphi(1)\psi(0)]\alpha^3 \\ &\quad +[3\varphi(1)\varphi(0) - \varphi(2)\varphi(1)]\alpha \qquad (3.1) \\ &= 0. \end{aligned}$$

By Lemma 2.4 and the assumptions $P(1) = 0$ and $P(2) \neq 0$, we have

$$\psi(1) = 2\psi(0), \quad \text{and} \quad P(2) = \psi(2) - 2\psi(1) + \psi(0) = \psi(2) - 3\psi(0) \neq 0.$$

Therefore, the coefficient of α^5 in (3.1) is

$$\psi(2)\psi(1) - 3\psi(1)\psi(0) = \psi(1)(\psi(2) - 3\psi(0)) = 2\psi(0)(\psi(2) - 3\psi(0)) \neq 0.$$

Hence, $J = 0$ for all but finitely many values of α. Let $\mathcal{A}_{22}$ denote the finite set of exceptional cases.

For $\alpha \notin \mathcal{A}_{22}$, the vectors $M_{p_\alpha}1$ and $M_{p_\alpha}^{*2}M_{p_\alpha}^3 1$ are linearly independent. Moreover, as discussed earlier, we have $\mathcal{S}E_0 = E_1$. Taking $\mathcal{A}_2 = \mathcal{A}_{21} \cup \mathcal{A}_{22}$, we conclude that the multiplication operator M_{p_α} is irreducible on $H^2(\omega, \delta)$ for all $\alpha \in (0, +\infty) \setminus \mathcal{A}_2$. That is, M_{p_α} is reducible on $H^2(\omega, \delta)$ only if $\alpha \in \mathcal{A}_2$. □

Remark 3.3. *From the above calculations, under the conditions $P(1) = 0$ and $P(2) \neq 0$, the functions $M_{p_\alpha}1$ and $M_{p_\alpha}^{*2}M_{p_\alpha}^3 1$ are linearly independent. However, $M_{p_\alpha}1$ and $M_{p_\alpha}^{*}M_{p_\alpha}^2 1$ are linearly dependent. Therefore, in the preceding proof we replace $M_{p_\alpha}^{*}M_{p_\alpha}^2 1$ by $M_{p_\alpha}^{*2}M_{p_\alpha}^3 1$.*

Example 3.4. *Assume $P(1) = P(2) = 0$ and $P(3) \neq 0$. Then there exists a countable set $\mathcal{A}_3$ such that for each $\alpha \in (0, +\infty)$, the multiplication operator M_{p_α} is reducible on the weighted Hardy space $H^2(\omega, \delta)$ only if $\alpha \in \mathcal{A}_3$.*

Proof. Define

$$\mathcal{A}_{31} = \{\alpha \in (0, +\infty) \ \mid \ \alpha^2 = \frac{\nabla\varphi(r-1) - \nabla\varphi(r-2)}{\nabla\psi(3) - \nabla\psi(2)} \ \text{ for some } \ r \geq 2\}.$$

For $\alpha \in (0, +\infty) \setminus \mathcal{A}_3$, we have for all $r \geq 2$ that

$$\nabla\varphi(r-1) - \nabla\varphi(r-2) \neq \alpha^2(\nabla\psi(3) - \nabla\psi(2)),$$

which implies

$$\nabla\varphi(r-1)+\alpha^2\nabla\psi(2)\neq\nabla\varphi(r-2)+\alpha^2\nabla\psi(3).$$

Thus, the relation $(r-1,2)\overset{\alpha}{\nsim}(r-2,3)$ holds for all $r\geq 2$. By Lemma 2.2, we obtain $\mathcal{S}E_r=E_{r+1}$ for all $r\geq 2$.

We now demonstrate that

$$\mathcal{S}E_0=E_1 \text{ and } \mathcal{S}E_1=E_2.$$

First, we prove $\mathcal{S}E_0=E_1$. Consider the functions $M_{p_\alpha}1$ and $M_{p_\alpha}^{*3}M_{p_\alpha}^4 1$. We have

$$\begin{aligned}M_{p_\alpha}1&=z+\alpha w,\\ M_{p_\alpha}^{*3}M_{p_\alpha}^4 1&=M_{z+\alpha w}^{*3}(z^4+4\alpha z^3w+6\alpha^2z^2w^2+4\alpha^3zw^3+\alpha^4w^4)\\ &=D_1z+D_2w,\end{aligned}$$

where

$$\begin{aligned}D_1&=\frac{\omega_4}{\omega_1}+12\alpha^2\frac{\omega_3}{\omega_1}\frac{\delta_1}{\delta_0}+18\alpha^4\frac{\omega_2}{\omega_1}\frac{\delta_2}{\delta_0}+4\alpha^6\frac{\delta_3}{\delta_0}\\ &=\varphi(3)\varphi(2)\varphi(1)+12\alpha^2\varphi(2)\varphi(1)\psi(0)+18\alpha^4\varphi(1)\psi(1)\psi(0)+4\alpha^6\psi(2)\psi(1)\psi(0),\\ D_2&=4\alpha\frac{\omega_3}{\omega_0}+18\alpha^3\frac{\omega_2}{\omega_0}\frac{\delta_2}{\delta_1}+12\alpha^5\frac{\omega_1}{\omega_0}\frac{\delta_3}{\delta_1}+\alpha^7\frac{\delta_4}{\delta_1}\\ &=4\alpha\varphi(2)\varphi(1)\varphi(0)+18\alpha^3\varphi(1)\varphi(0)\psi(1)+12\alpha^5\varphi(0)\psi(2)\psi(1)+\alpha^7\psi(3)\psi(2)\psi(1).\end{aligned}$$

We show that $M_{p_\alpha}1$ and $M_{p_\alpha}^{*3}M_{p_\alpha}^4 1$ are linearly independent. Once this is established, we have $\dim\mathcal{S}E_0\geq 2$. Since $\mathcal{S}E_0\subseteq E_1$ and $\dim E_1=2$, it follows that $\mathcal{S}E_0=E_1$.

Suppose, for contradiction, that $M_{p_\alpha}1$ and $M_{p_\alpha}^{*3}M_{p_\alpha}^4 1$ are linearly dependent. Then

$$J:=\begin{vmatrix}1&\alpha\\ D_1&D_2\end{vmatrix}=0.$$

Computing the determinant yields

$$\begin{aligned}J=&\,\psi(2)\psi(1)(\psi(3)-4\psi(0))\alpha^7+6\psi(2)\psi(1)(2\varphi(0)-3\varphi(1))\alpha^5\\ &+6\varphi(1)\varphi(0)(3\psi(1)-2\varphi(2))\alpha^3+\varphi(2)\varphi(1)(4\varphi(0)-\varphi(3))\alpha=0.\end{aligned}$$

By Lemma 2.4 and the assumptions $P(1)=P(2)=0$ and $P(3)\neq 0$, we have

$$\psi(1)=2\psi(0),\quad \psi(2)=3\psi(0),$$

and moreover,

$$P(3)=\psi(3)-2\psi(2)+\psi(1)=\psi(3)-4\psi(0)\neq 0.$$

Hence, the coefficient of α^7 is nonzero. Therefore, the equation has at most seven solutions. Let $\mathcal{A}_{32}$ be this finite set. Then for $\alpha\notin\mathcal{A}_{32}$, $M_{p_\alpha}1$ and $M_{p_\alpha}^{*3}M_{p_\alpha}^4 1$ are linearly independent, which leads to that $\mathcal{S}E_0=E_1$.

Next we show $\mathcal{S}E_1=E_2$. Consider the functions

$$M_{p_\alpha}z=z^2+\alpha zw,\quad M_{p_\alpha}w=zw+\alpha w^2,$$

and

$$\begin{aligned}M_{p_\alpha}^{*2}M_{p_\alpha}^3 w&=M_{z+\alpha w}^{*2}(z^3w+3\alpha z^2w^2+3\alpha^2zw^3+\alpha^3w^4)\\ &=C_1z^2+C_2zw+C_3w^2,\end{aligned}$$

where

$$\begin{aligned}
C_1 &= 2\alpha\frac{\omega_3}{\omega_2}\frac{\delta_1}{\delta_0} + 3\alpha^3\frac{\delta_2}{\delta_0}\\
&= 2\alpha\varphi(2)\psi(0) + 3\alpha^3\psi(1)\psi(0),\\
C_2 &= \frac{\omega_3}{\omega_1} + 6\alpha^2\frac{\omega_2}{\omega_0}\frac{\delta_2}{\delta_1} + 3\alpha^4\frac{\delta_3}{\delta_1}\\
&= \varphi(2)\varphi(1) + 6\alpha^2\varphi(1)\psi(1) + 3\alpha^4\psi(2)\psi(1),\\
C_3 &= 3\alpha\frac{\omega_2}{\omega_0} + 6\alpha^2\frac{\omega_1}{\omega_0}\frac{\delta_3}{\delta_2} + \alpha^5\frac{\delta_4}{\delta_2}\\
&= 3\alpha\varphi(1)\varphi(0) + 6\alpha^3\varphi(0)\psi(2) + \alpha^5\psi(3)\psi(2).
\end{aligned}$$

If $M_{p_\alpha}z$, $M_{p_\alpha}w$, and $M_{p_\alpha}^{*2}M_{p_\alpha}^3 w$ are linearly independent, then $\dim \mathcal{S}E_1 \geq 3$. Since $\mathcal{S}E_1 \subseteq E_2$ and $\dim E_2 = 3$, it follows that $\mathcal{S}E_1 = E_2$.

Suppose they are linearly dependent. Then the coefficient determinant satisfies

$$\widehat{J} := \begin{vmatrix} 1 & \alpha & 0 \\ 0 & 1 & \alpha \\ C_1 & C_2 & C_3 \end{vmatrix} = 0.$$

Expanding along the first row gives

$$\begin{aligned}
\widehat{J} &= C_3 - \alpha C_2 + \alpha^2 C_1\\
&= [3\varphi(0)\varphi(1) - \varphi(1)\varphi(2)]\alpha + [6\varphi(0)\psi(2) - 6\varphi(1)\psi(1) + 2\varphi(2)\psi(0)]\alpha^3\\
&\quad + [\psi(3)\psi(2) - 3\psi(1)\psi(2) + 3\psi(0)\psi(1)]\alpha^5 = 0.
\end{aligned}$$

Again, the coefficient of α^5 is nonzero. Thus, $\widehat{J} \neq 0$ except for finitely many α. Let $\mathcal{A}_{33}$ be this finite set. Then for $\alpha \notin \mathcal{A}_{33}$, $M_{p_\alpha}z$, $M_{p_\alpha}w$, and $M_{p_\alpha}^{*2}M_{p_\alpha}^3 w$ are linearly independent. Furthermore, we have $\mathcal{S}E_1 = E_2$.

Let $\mathcal{A}_3 = \mathcal{A}_{31} \cup \mathcal{A}_{32} \cup \mathcal{A}_{33}$ be a finite set. Then for all $\alpha \in (0, +\infty) \setminus \mathcal{A}_3$, we have $\mathcal{S}E_r = E_{r+1}$ for all $r \geq 0$. By Lemma 2.3, the multiplication operator M_{p_α} is irreducible on the weighted Hardy space $H^2(\omega, \delta)$ for $\alpha \in (0, +\infty) \setminus \mathcal{A}_3$. That is, M_{p_α} is reducible on the weighted Hardy space $H^2(\omega, \delta)$ only if $\alpha \in \mathcal{A}_3$. □

4. Proofs of Theorem 1.2 and 1.5

4.1. **Some preparations.** For the proof of Theorem 1.5, we establish some lemmas.

The following lemma establishes that the $(k+1)$-th backward difference of a degree-k polynomial is identically zero, mirroring the analogous result for forward differences (see [8, Section 5.3], for example).

Lemma 4.1. *Let f be a polynomial of degree at most k. Then its $(k+1)$-th backward difference vanishes; that is,*

$$\nabla^{k+1} f(y) := \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} f(y-i) = 0. \tag{4.1}$$

Proof. For $k = 0$, we have

$$\nabla f(y) = \sum_{i=0}^{1} (-1)^i \binom{1}{i} f(y-i) = f(y) - f(y-1),$$

and for $k = 1$,

$$\begin{aligned}\nabla^2 f(y) &= \sum_{i=0}^{2}(-1)^i\binom{2}{i}f(y-i)\\ &= f(y)-f(y-1)-(f(y-1)-f(y-2))\\ &= \nabla f(y)-\nabla f(y-1)=\nabla(\nabla f)(y).\end{aligned}$$

By induction, we can show that

$$\nabla^{k+1}f(y)=\sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}f(y-i)=\nabla(\nabla^k f)(y);$$

that is, $\nabla^{k+1}f=\nabla(\nabla^k f)$.

Using this property and the linearity of the backward difference operator ∇, by induction we have $\nabla^{k+1}f=0$ for every polynomial of degree at most k. The proof is complete. □

For integers k and n_0 with $2 \le k \le n_0$, define $S(k, n_0)$ by

$$S(k,n_0)=\sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}\prod_{m=0}^{k-1}\psi(n_0-i-m).$$

Recall that in the above equation $\psi(-1)=\psi(-2)=\cdots=0$.

Lemma 4.2. *Suppose there exists $n_0 \ge 2$ such that*

$$P(1)=P(2)=\cdots=P(n_0-1)=0 \quad \textit{and} \quad P(n_0)\neq 0.$$

Then

$$S(k,n_0)=\psi(0)^{k-1}\frac{n_0!}{(n_0-k+1)!}P(n_0),$$

where $P(n_0)=\psi(n_0)-(n_0+1)\psi(0)$.

Proof. Define

$$Q(i)=\prod_{m=0}^{k-1}\psi(n_0-i-m), \quad i\ge 0.$$

For $i=0$, we compute

$$\begin{aligned}Q(0)&=\prod_{m=0}^{k-1}\psi(n_0-m)\\ &=\psi(n_0)\prod_{m=1}^{k-1}\psi(n_0-m)\\ &=\psi(n_0)\prod_{m=1}^{k-1}[(n_0-m+1)\psi(0)]\\ &=\psi(n_0)\psi(0)^{k-1}\prod_{m=1}^{k-1}(n_0-m+1).\end{aligned}$$

For $1\le i\le k+1$, since $n_0-i-m\le n_0-1$ for all $0\le m\le k-1$, we have

$$Q(i)=\prod_{m=0}^{k-1}\psi(n_0-i-m)=\prod_{m=0}^{k-1}[(n_0-i-m+1)\psi(0)]=\psi(0)^k\prod_{m=0}^{k-1}(n_0-i-m+1).$$

Thus,

$$\begin{aligned}S(k,n_0) &= \sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}Q(i)\\ &= Q(0)+\sum_{i=1}^{k+1}(-1)^i\binom{k+1}{i}Q(i)\\ &= \psi(n_0)\psi(0)^{k-1}\prod_{m=1}^{k-1}(n_0-m+1)+\sum_{i=1}^{k+1}(-1)^i\binom{k+1}{i}\psi(0)^k\prod_{m=0}^{k-1}(n_0-i-m+1).\end{aligned}$$

The falling factorial is defined as $y^{\underline{k-1}} = y(y-1)\cdots(y-k+2)$ for $y\in\mathbb{R}$. This is a polynomial of degree $k-1$ in y. Note that the first term can be rewritten as

$$\psi(n_0)\psi(0)^{k-1}\prod_{m=1}^{k-1}(n_0-m+1) = \psi(n_0)\psi(0)^{k-1}n_0^{\underline{k-1}}.$$

For the second term, let $y_0 = n_0+1$,

$$\sum_{i=1}^{k+1}(-1)^i\binom{k+1}{i}\psi(0)^k\prod_{m=0}^{k-1}(n_0-i-m+1) = \sum_{i=1}^{k+1}(-1)^i\binom{k+1}{i}\psi(0)^k(y_0-i)^{\underline{k}},$$

Thus,

$$S(k,n_0) = \psi(n_0)\psi(0)^{k-1}n_0^{\underline{k-1}}+\psi(0)^k\sum_{i=1}^{k+1}(-1)^i\binom{k+1}{i}(y_0-i)^{\underline{k}}.$$

Consider the sum

$$\sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}(y_0-i)^{\underline{k}},$$

By Lemma 4.1, we have the $(k+1)$-th backward difference of a degree-k polynomial is zero, we have

$$\sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}(y_0-i)^{\underline{k}} = 0.$$

Hence,

$$\sum_{i=1}^{k+1}(-1)^i\binom{k+1}{i}(y_0-i)^{\underline{k}} = -\binom{k+1}{0}(y_0-0)^{\underline{k}} = -y_0^{\underline{k}}.$$

Substituting $y_0 = n_0+1$ gives

$$\sum_{i=1}^{k+1}(-1)^i\binom{k+1}{i}\prod_{m=0}^{k-1}(n_0-i-m+1) = -(n_0+1)^{\underline{k}}.$$

Therefore,

$$\begin{aligned}S(k,n_0) &= \psi(n_0)\psi(0)^{k-1}n_0^{\underline{k-1}}-\psi(0)^k(n_0+1)^{\underline{k}}\\ &= \psi(0)^{k-1}n_0^{\underline{k-1}}[\psi(n_0)-(n_0+1)\psi(0)]\\ &= \psi(0)^{k-1}\frac{n_0!}{(n_0-k+1)!}P(n_0),\end{aligned}$$

where the last equality follows from $P(n_0) = \psi(n_0)-(n_0+1)\psi(0)$. The proof is complete. $\square$

4.2. **Proofs of Theorem 1.2 and 1.5.** For convenience, we restate Theorem 1.5 as follows.

Theorem 4.3. *Suppose there exists $n_0 \geq 2$ such that*

$$P(1) = P(2) = \cdots = P(n_0 - 1) = 0 \quad \text{and} \quad P(n_0) \neq 0.$$

Then for $\alpha \in (0, +\infty)$, $M_{z+\alpha w}$ is reducible on the weighted Hardy space $H^2(\omega, \delta)$ for at most a countable subset $\mathcal{A}_{n_0}$ of α.

Proof. For the cases $n_0 = 2$ and $n_0 = 3$, we refer to Examples 3.2 and 3.4, respectively. In general, let $\alpha \in (0, +\infty)$. By Lemma 2.3, to show $M_{z+\alpha w}$ is reducible on the weighted Hardy space $H^2(\omega, \delta)$ for at most a countable subset of α, it suffices to prove for all such α outside a countable set, the equality $\mathcal{S}E_r = E_{r+1}$ holds for every r.

We first establish this for all α outside a countable set, and all $r \geq n_0 - 1$. To see this, define the set

$$\mathcal{A}_{n_0,1} = \{\alpha \in \mathbb{R} \mid \alpha^2 = \frac{\nabla\varphi(r - n_0 + 2) - \nabla\varphi(r - n_0 + 1)}{\nabla\psi(n_0) - \nabla\psi(n_0 - 1)} \ \text{ for some } \ r \geq n_0 - 1\}.$$

For every $\alpha \notin \mathcal{A}_{n_0,1}$, we have for all $r \geq n_0 - 1$ that

$$\nabla\varphi(r - n_0 + 2) - \nabla\varphi(r - n_0 + 1) \neq \alpha^2(\nabla\psi(n_0) - \nabla\psi(n_0 - 1)),$$

which implies

$$\nabla\varphi(r - n_0 + 2) + \alpha^2\nabla\psi(n_0 - 1) \neq \nabla\varphi(r - n_0 + 1) + \alpha^2\nabla\psi(n_0).$$

Consequently, the relation $(r - n_0 + 2, n_0 - 1) \overset{\alpha}{\not\approx} (r - n_0 + 1, n_0)$ holds for all $r \geq n_0 - 1$. By Lemma 2.2, it follows that $\mathcal{S}E_r = E_{r+1}$ for all $r \geq n_0 - 1$.

We now turn to the case where $r < n_0 - 1$. That is, we will invoke Lemma 4.2 to prove that $SE_{n_0-k} = E_{n_0-k+1}$ for each $2 \leq k \leq n_0$.

Consider the operators

$$M_{p_\alpha} \quad \text{and} \quad M_{p_\alpha}^{*k} M_{p_\alpha}^{k+1},$$

acting on the monomials $z^i w^j$ (with $i + j = n_0 - k$) and w^{n_0-k}, respectively. By direct computations, we have

$$\begin{aligned}
M_{p_\alpha} z^{n_0-k} &= z^{n_0-k+1} + \alpha z^{n_0-k} w, \\
M_{p_\alpha} z^{n_0-k-1} w &= z^{n_0-k} w + \alpha z^{n_0-k-1} w^2, \\
&\ \ \vdots \\
M_{p_\alpha} w^{n_0-k} &= z w^{n_0-k} + \alpha w^{n_0-k+1}, \\
M_{p_\alpha}^{*k} M_{p_\alpha}^{k+1} w^{n_0-k} &= M_{z+\alpha w}^{*k} \sum_{j=0}^{k+1} \binom{k+1}{j} \alpha^j z^{k+1-j} w^{n_0-k+j}, \\
&\equiv D_1 z^{n_0-k+1} + D_2 z^{n_0-k} w + \cdots + D_{n_0-k+2} w^{n_0-k+1},
\end{aligned}$$

where $D_1, D_2, \cdots, D_{n_0-k+2}$ are the corresponding coefficients. If these $n_0 - k + 2$ functions are linearly independent, then

$$\dim \mathcal{S}E_{n_0-k} \geq n_0 - k + 2 = \dim E_{n_0-k+1}.$$

Since $\mathcal{S}E_{n_0-k} \subseteq E_{n_0-k+1}$ and $\dim E_{n_0-k+1} = n_0 - k + 2$, we conclude that $\mathcal{S}E_{n_0-k} = E_{n_0-k+1}$.

To reach a contradiction, we assume these functions are linearly dependent. Then their coefficient determinant necessarily becomes zero. That is,

$$J := \begin{vmatrix} 1 & \alpha & 0 & \cdots & 0 & 0 \\ 0 & 1 & \alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \alpha \\ D_1 & D_2 & D_3 & \cdots & D_{n_0-k+1} & D_{n_0-k+2} \end{vmatrix} = 0.$$

Observe that the highest power of α in $D_1, D_2, \ldots, D_{n_0-k+2}$ arises from the operator $M_{\alpha w}^{*k}$ applied to the sum

$$\sum_{j=0}^{k+1} \binom{k+1}{j} \alpha^j z^{k+1-j} w^{n_0-k+j}.$$

The leading term α^{2k+1} in J is determined by the highest-order coefficients in α from $D_1, D_2, \ldots, D_{n_0-k+2}$.

Expanding the determinant J along the first row gives

$$J = \sum_{i=0}^{n_0-k+1} (-1)^i \alpha^i D_{n_0-k+2-i},$$

a polynomial in α of degree at most $2k+1$. The coefficient of α^{2k+1} is

$$\widehat{S}(k, n_0) = \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} \frac{\delta_{n_0+1-i}}{\delta_{n_0-k+1-i}} = \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} \prod_{m=0}^{k-1} \psi(n_0 - i - m).$$

By Lemma 4.1, we have

$$\widehat{S}(k, n_0) = \psi(0)^{k-1} \cdot \frac{n_0!}{(n_0-k+1)!} \cdot P(n_0),$$

where $P(n_0) \neq 0$. Thus $\widehat{S}(k, n_0) \neq 0$. So the equation $J = 0$ is a nonzero polynomial equation of degree $2k+1$, and its solution set $\mathcal{A}_{n_0,k}$ is finite. For every $\alpha \notin \mathcal{A}_{n_0,k}$, we obtain $\mathcal{S}E_{n_0-k} = E_{n_0-k+1}$ for $2 \leq k \leq n_0$.

Now define

$$\mathcal{A}_{n_0} = \bigcup_{k=1}^{n_0} \mathcal{A}_{n_0,k}.$$

Since each $\mathcal{A}_{n_0,k}$ is finite, $\mathcal{A}_{n_0}$ is a countable set. For every $\alpha \in (0, +\infty) \setminus \mathcal{A}_{n_0}$, we have $\mathcal{S}E_r = E_{r+1}$ for all $r \geq 0$. Therefore, by Lemma 2.3 the multiplication operator M_{p_α} is irreducible on the weighted Hardy space $H^2(\omega, \delta)$ for this α. The proof of Theorem 4.3 is complete. □

We now give the proof of Theorem 1.2.

Proof of Theorem 1.2. By Proposition 2.5, $P(n)$ cannot be identically zero for all $n \geq 1$. There are two cases under consideration.

Case 1. $P(1) \neq 0$. In this case, by Example 3.1 we conclude that for $\alpha \in (0, +\infty)$, $M_{z+\alpha w}$ is reducible on the weighted Hardy space $H^2(\omega, \delta)$ for at most a countable set of α.

Case 2. There exists $n_0 \geq 2$ such that

$$P(1) = P(2) = \cdots = P(n_0 - 1) = 0 \quad \text{and} \quad P(n_0) \neq 0.$$

Then, by Theorem 4.3, for $\alpha \in (0, +\infty)$, $M_{z+\alpha w}$ is reducible on the weighted Hardy space $H^2(\omega, \delta)$ for at most a countable subset of α.

The proof of Theorem 1.2 is complete. $\square$

5. Some examples

5.1. **The countable set in Theorem 1.2 being empty.** In this section, we will show that for arbitrary α, the operator $M_{z+\alpha w}$ is irreducible on $H^2(\mathbb{D}) \otimes L_a^2(\mathbb{D})$.

Proposition 5.1. *For $\alpha > 0$, the operator $M_{z+\alpha w}$ is irreducible on $H^2(\mathbb{D}) \otimes L_a^2(\mathbb{D})$.*

The proof of this proposition relies on the following lemma.

Lemma 5.2. *For all $\alpha > 0$, we have $\mathcal{S}E_r = E_{r+1}, r \geq 0$.*

Proof. Recall that $T = [M^*_{z+\alpha w}, M_{z+\alpha w}]$. By Lemma 2.1, we have

$$Tz^m w^n = \lambda_\alpha(m,n) z^m w^n,$$

where

$$\lambda_\alpha(m,n) = \begin{cases} 1 + \frac{1}{2}\alpha^2, & \text{if } m = 0,\ n = 0, \\ \frac{1}{2}\alpha^2, & \text{if } m \geq 1,\ n = 0, \\ 1 + \dfrac{1}{(n+1)(n+2)}\alpha^2, & \text{if } m = 0,\ n \geq 1, \\ \dfrac{1}{(n+1)(n+2)}\alpha^2, & \text{if } m \geq 1,\ n \geq 1. \end{cases} \tag{5.1}$$

From (5.1), we obtain that for all $n \geq 1$,

$$\frac{1}{(n+1)(n+2)}\alpha^2 \neq \frac{1}{2}\alpha^2,$$

which implies $(r+1, 0) \overset{\alpha}{\nsim} (r, 1)$ for all $r \geq 1$. Hence, by Lemma 2.2, it follows that $\mathcal{S}E_r = E_{r+1}$.

It remains to show $\mathcal{S}E_0 = E_1$. For this, note that $\mathcal{S}E_0 \subseteq E_1$. To prove the reverse inclusion $\mathcal{S}E_0 \supseteq E_1$, consider the operators

$$M_{P_\alpha},\ M^*_{P_\alpha} M^2_{P_\alpha},\ M^{*2}_{P_\alpha} M^3_{P_\alpha} \in \mathcal{S}.$$

By direct computations, we have

$$M_{P_\alpha} 1 = z + \alpha w, \tag{5.2}$$

$$M^*_{P_\alpha} M^2_{P_\alpha} 1 = (1+\alpha^2) z + \alpha(2 + \frac{2}{3}\alpha^2) w, \tag{5.3}$$

$$M^{*2}_{P_\alpha} M^3_{P_\alpha} 1 = (1 + 3\alpha^2 + \alpha^4) z + \alpha(3 + 4\alpha^2 + \frac{1}{2}\alpha^4) w. \tag{5.4}$$

In view of (5.2) and (5.3), the determinant of the coefficient matrix is as follows:

$$\begin{vmatrix} 1 & 1 \\ 1+\alpha^2 & 2 + \frac{2}{3}\alpha^2 \end{vmatrix} = 1 - \frac{1}{3}\alpha^2 \neq 0.$$

Consequently, for $\alpha^2 \neq 3$, we conclude $\mathcal{S}E_0 = E_1$.

Now suppose $\alpha^2 = 3$. Using (5.2) and (5.4), we find

$$\begin{vmatrix} 1 & 1 \\ 1 + 3\alpha^2 + \alpha^4 & 3 + 4\alpha^2 + \frac{1}{2}\alpha^4 \end{vmatrix} = \frac{1}{2}\alpha^4 - \alpha^2 - 2 \neq 0,$$

i.e., $\alpha^2 \neq 1+\sqrt{5}$ and $\alpha^2 \neq 1-\sqrt{5}$. Since the vectors $M_{P_\alpha}1$ and $M_{P_\alpha}^{*2}M_{P_\alpha}^3 1$ are linearly independent when $\alpha^2 = 3$, it follows that $\dim \mathcal{S}E_0 = 2$. Therefore, $\mathcal{S}E_0 = E_1$. □

Combining Lemma 2.3 with Lemma 5.2 immediately gives Proposition 5.1.

5.2. **The countable set in Theorem 1.2 being a singleton.** Let $\omega = \{\omega_n\}_{n\geq 0}$ be a sequence of positive numbers satisfying

$$R := \liminf_{n\to\infty}(\omega_n)^{1/n} > 0.$$

In the case when $\omega = \delta$, the space $H^2(\omega,\delta)$ reduces to the weighted Hardy space $H^2(\omega,\omega)$. Here, we define

$$\widetilde{P}(n) = \nabla\psi(n) - \psi(0).$$

Theorem 5.3. *Suppose $\widetilde{P}(n) \neq 0$ for all $n \geq 1$. Then the multiplication operator M_{p_α} is reducible on $H^2(\omega,\omega)$ if and only if $\alpha = 1$.*

Proof. Observe that in the space $H^2(\omega,\omega)$, we have

$$Tz^m w^n = \lambda_\alpha(m,n) z^m w^n = (\nabla\psi(m) + \alpha^2\nabla\psi(n)) z^m w^n$$

for all $m, n \geq 0$. For $\alpha \neq 1$, we have

$$\nabla\psi(r+1) - \psi(0) \neq \alpha^2(\nabla\psi(r+1) - \psi(0)) \quad \text{for all } r \geq 0,$$

which yields

$$\nabla\psi(r+1) + \alpha^2\psi(0) \neq \psi(0) + \alpha^2\nabla\psi(r+1) \quad \text{for all } r \geq 0,$$

which implies $(r+1,0) \overset{\alpha}{\nsim} (0,r+1)$ for all $r \geq 0$.

By Lemma 2.2, it follows that $\mathcal{S}E_r = E_{r+1}$ for all $r \geq 0$. Then applying Lemma 2.3, for $\alpha \neq 1$, we conclude that the multiplication operator M_{p_α} is irreducible on $H^2(\omega,\omega)$. In other words, M_{p_α} is reducible on $H^2(\omega,\omega)$ only if $\alpha = 1$.

It remains to prove that M_{z+w} is reducible. For this, denote the symmetric and antisymmetric subspaces of $H^2(\omega,\omega)$ by H_s and H_a, respectively. That is,

$$H_s = \{f \in H^2(\omega,\omega) : f(z,w) = f(w,z)\},$$

and

$$H_a = \{f \in H^2(\omega,\omega) : f(z,w) = -f(w,z)\}.$$

Clearly, $H^2(\omega,\omega) = H_s \oplus H_a$. It is routine to check that both H_s and H_a are invariant under both M_{z+w} and M_{z+w}^*. Therefore, the multiplication operator M_{z+w} is reducible on $H^2(\omega,\omega)$. The proof is complete. □

From now on, $\omega = \{\omega_n\}_{n\geq 0}$ is such that

$$R := \liminf_{n\to\infty}(\omega_n)^{1/n} = 1.$$

For such ω, the weighted Hardy space $H^2_\omega(\mathbb{D})$ consists of all holomorphic functions $f(z) = \sum_{n\geq 0} a_n z^n$ on $\mathbb{D}$ satisfying

$$\|f\|^2 = \sum_{n\geq 0} \omega_n |a_n|^2 < \infty.$$

The weighted Hardy space $H^2_\omega(\mathbb{D}^2)$ can be identified with the tensor product $H^2_\omega(\mathbb{D}) \otimes H^2_\omega(\mathbb{D})$.

For $\rho > 0$, put

$$\omega = \left\{ \omega_n = \frac{n!\Gamma(\rho)}{\Gamma(\rho+n)} \right\}_{n\geq 0}.$$

It can be verified that $R = \liminf_{n\to\infty} (\omega_n)^{1/n} = 1$. Rewrite $\mathcal{K}_\rho(\mathbb{D}^2) = H^2_\omega(\mathbb{D}^2)$, which constitutes a parameterized family of function spaces. Some special cases are noteworthy. When $\rho = 1$, we obtain the classical Hardy space $\mathcal{K}_1(\mathbb{D}^2) = H^2(\mathbb{D}^2)$. For $\rho = 2$, the space coincides with the standard Bergman space $\mathcal{K}_2(\mathbb{D}^2) = L^2_a(\mathbb{D}^2)$. For $\rho > 1$, the space $\mathcal{K}_\rho(\mathbb{D}^2)$ is known as the weighted Bergman space on $\mathbb{D}^2$. In this context, the norm of a function $f \in \mathcal{K}_\rho(\mathbb{D}^2)$ is given by the L^2 norm with respect to the weighted area measure $(1-|z|^2)^\rho(1-|w|^2)^\rho dA(z)dA(w)$ on $\mathbb{D}^2$.

Next, consider the weighted Hardy space $H^2_\omega(\mathbb{D}^2)$ equipped with the weight sequence

$$\omega = \{\omega_n = (n+1)^\rho\}_{n\geq 0}.$$

We call this space the weighted Dirichlet space, denoted by $\mathcal{D}_\rho(\mathbb{D}^2)$. When $\rho = 1$, this reduces to the classical Dirichlet space $\mathcal{D}(\mathbb{D}^2)$ (see [22]).

Lemma 5.4. *For $\rho > 0$, let $\{\omega_n\}$ be defined by one of the following,*
(1) $\omega_n = \frac{n!\Gamma(\rho)}{\Gamma(\rho+n)}$,
(2) $\omega_n = (n+1)^\rho$.
Then the sequence $\{\psi(k)\}_{k\geq 0}$ satisfies $\widetilde{P}(n) \neq 0$ for all $n \geq 1$.

Proof. We verify the condition $\widetilde{P}(n) \neq 0$ separately for two cases.

Case (1). Let $\omega_n = \dfrac{n!\Gamma(\rho)}{\Gamma(\rho+n)}$. We show that for all $\rho > 0$ and $n \geq 1$,

$$\nabla\psi(n) - \psi(0) = \frac{\rho-1}{(\rho+n)(\rho+n-1)} - \frac{1}{\rho} \neq 0.$$

If $0 < \rho < 1$, then $\rho - 1 < 0$ and the denominator is positive. Thereby the expression is negative.
If $\rho = 1$, $\nabla\psi(n) - \psi(0) = -1 \neq 0$.
If $\rho > 1$, suppose for contradiction that

$$\frac{\rho-1}{(\rho+n)(\rho+n-1)} = \frac{1}{\rho}.$$

This simplifies to the equation $n(2\rho + n - 1) = 0$. However, the equation has no positive integer solutions for n, yielding the desired contradiction.

Case (2). Let $\omega_n = (n+1)^\rho$, $\rho > 0$. Define $f(x) = \left(\dfrac{x+2}{x+1}\right)^\rho$ for $x \geq 0$. Then

$$f'(x) = \rho\left(\frac{x+2}{x+1}\right)^{\rho-1} \cdot \frac{-1}{(x+1)^2} < 0,$$

which implies that f is strictly decreasing. Hence, for $n \geq 1$,

$$\nabla\psi(n) = f(n) - f(n-1) = \left(\frac{n+2}{n+1}\right)^\rho - \left(\frac{n+1}{n}\right)^\rho < 0.$$

Therefore,

$$\nabla\psi(n) - \psi(0) = \left(\frac{n+2}{n+1}\right)^\rho - \left(\frac{n+1}{n}\right)^\rho - 2^\rho < 0.$$

In either case, we obtain $\widetilde{P}(n) \neq 0$ for all $n \geq 1$, concluding the proof. $\square$

The following results follow from combining Theorem 5.3 and Lemma 5.4.

Corollary 5.5. *For $\alpha > 0$, M_{p_α} is reducible on $\mathcal{K}_\rho(\mathbb{D}^2)$ if and only if $\alpha = 1$.*

In particular, when $\rho = 1$, M_{p_α} is reducible on $H^2(\mathbb{D}^2)$ if and only if $\alpha = 1$; when $\rho = 2$, M_{p_α} is reducible on $L_a^2(\mathbb{D}^2)$ if and only if $\alpha = 1$. (see [5, 28]).

Corollary 5.6. *For $\alpha > 0$, M_{p_α} is reducible on $\mathcal{D}_\rho(\mathbb{D}^2)$ if and only if $\alpha = 1$.*

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