PARABOLIC RESTRICTIONS AND DOUBLE DEFORMATIONS OF WEIGHT MULTIPLICITIES

CÉDRIC LECOUVEY

Abstract. We introduce some (p,q)-deformations of the weight multiplicities for the representations of any simple Lie algebra $\mathfrak{g}$ over the complex numbers. This is done by associating the indeterminate q to the positive roots of a parabolic subsystem of $\mathfrak{g}$ and the indeterminate p to the remaining positive roots. When $p=q$, we just recover the usual Lusztig q-analogues of weight multiplicities. We then study the positivity of the coefficients in these double deformations. In particular, the positivity holds when $p=1$ in which case the polynomials have a natural algebraic interpretation in terms of a parabolic Brylinski filtration. For the parabolic restriction from type C to type A, this positivity result was conjectured by Lee. We also establish this positivity, in any finite type and for any p, for a stabilized version of our double deformation. In addition, we study the double deformation obtained by replacing the pair (p,q) by $(p+1,q+1)$, show it has nonnegative coefficients and admits a combinatorial description in terms of crystals.

1. Introduction

In this article, we introduce a double deformation of the weight multiplicities for the representations of any simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ depending on the choice of a parabolic root system and two indeterminates p and q. The deformation is constructed from the (p,q)-Kostant partition function defined by the series

$$\prod_{\alpha\in R_+\setminus\overline{R}_+}\frac{1}{1-pe^{\alpha}}\prod_{\alpha\in\overline{R}_+}\frac{1}{1-qe^{\alpha}}=\sum_{\beta\in Q_+}\mathrm{P}_{p,q}^{\overline{R}_+}(\beta)e^{\beta}$$

by mimicking Kostant's classical weight formula (see (4) for the precise definition). Here R_+ is the set of positive roots of $\mathfrak{g}$ and $\overline{R}_+$ the set of positive roots corresponding to a parabolic subsystem, that is the set of positive roots associated to a Levi subalgebra $\overline{\mathfrak{g}}$ of $\mathfrak{g}$. For any dominant weight ν and any weight μ (ν and μ are weights for $\mathfrak{g}$), we so get polynomials $K_{\nu,\mu}^{\overline{R}_+}(p,q)$ in $\mathbb{Z}[p,q]$ such that $K_{\nu,\mu}^{\overline{R}_+}(q,q)$ coincides with the ordinary Lusztig q-analogue [18]. In particular $K_{\nu,\mu}^{\overline{R}_+}(1,1)$ gives the dimension of the weight space μ in the irreducible $\mathfrak{g}$-module $V(\nu)$ of highest weight ν. The polynomial $K_{\nu,\mu}^{\overline{R}_+}(p,q)$ depends a priori on the choice of a parabolic root system but for simplicity we will drop the superscript $\overline{R}_+$ and assume that this parabolic root system is fixed once for all in each statement.

In Section 3, we give a decomposition of the polynomial $K_{\nu,\mu}(p,q)$ in terms of p-branching multiplicities corresponding to the restriction from $\mathfrak{g}$ to $\overline{\mathfrak{g}}$ and ordinary Lusztig q-weight multiplicities for $\overline{\mathfrak{g}}$ (Theorem 3.1). This permits to study the positivity of the coefficients of the polynomials $K_{\nu,\mu}(p,q)$. We establish that $K_{\nu,\mu}(1,q)$ and $K_{\nu,\mu}(0,q)$ belong to $\mathbb{N}[q]$ when ν and μ are dominant weights. This proves as a special case Conjecture 1.4 proposed by Lee in [17] for the parabolic restriction from type C_n to type A_{n-1}. In fact a proof of this conjecture was announced in [17] and next established in [4] (essentially at the same time as this article appeared) together with the particular cases of the Levi restrictions from types B_n and D_n to type A_{n-1}. Our result here holds in full generality for any parabolic restriction. Also, we show that the polynomials $K_{\nu,\mu}(1,q)$ coincide with the jump polynomials corresponding to the parabolic Brylinski-Kostant filtration of the weight space μ in $V(\nu)$ associated to a principal nilpotent element $\overline{e}$ in $\overline{\mathfrak{g}}$. It is interesting here to also mention that, in the classical cases and for their parabolic root subsystems of type A_{n-1}, the polynomials $K_{\nu,\mu}(1,q)$ coincide with some row shaped one-dimensional sums (see [4] for the related definitions, precise statements and proofs).

In Section 4, we explain how one can relax the hypothesis for μ to be dominant and prove, by using arguments due to Panyushev [22], that the polynomial $K_{\nu,\mu}(p+1,q+1)$ belongs to $\mathbb{N}[p,q]$ for any dominant weight λ and any weight μ (Theorem 4.1). In Section 5, we get some positivity results for stabilized versions of the polynomials $K_{\nu,\mu}(p,q)$. In the classical types and for the ordinary Lusztig q-analogues, such stabilized versions were defined in [12] and [13] and proved to coincide with one-dimensional sums in [15]. The stabilization was then obtained by translating the dominant weights ν and μ by a multiple of the fundamental weight ω_n. Here we generalize the construction in both directions: first this is done for our two parameters deformations and next we get stabilized polynomials for any parabolic subsystem of a given root system (Theorem 5.4). Finally, in Section 6, we examine the question of the combinatorial description of the weight multiplicities $K_{\nu,\mu}(1,q)$ in terms of crystal graphs. In particular, this description is easy when $\overline{\mathfrak{g}}$ is maximal of type A, that is its Dynkin diagram is obtained by removing in that of $\mathfrak{g}$ the unique node for which $\overline{\mathfrak{g}}$ becomes of type A. We also give a combinatorial description of the polynomials $K_{\nu,\mu}(p+1,q+1)$ by introducing a polynomial charge statistics on the vertices of the crystal $B(\nu)$ associated to the dominant weight ν. This construction is reminiscent to Lascoux-Schützenberger's charge statistics but in our setting, one associates a polynomial to each vertex of $B(\nu)$ (for example a semistandard tableau in type A) and not only a monomial. In particular the polynomials $K_{\nu,\mu}(q+1,q+1) = K_{\nu,\mu}(q+1)$, which are just the Lusztig q-analogues evaluated in $q+1$ instead of q, admit simple combinatorial descriptions for any weight μ even if their algebraic interpretation is yet unclear. As far as we are aware this combinatorial description is new. We conclude the section by proposing some (p,q)-analogues of the Hall-Littlewood polynomials.

2. Double deformation of weight multiplicities

In this Section, we recall some classical results on root systems and the representation theory of the Lie algebras over $\mathbb{C}$. We refer the reader to [1], [5], or [8] for a detailed exposition. Consider such a finite-dimensional simple algebra $\mathfrak{g}$ and assume its root system R is realized in the Euclidean space E. Its Dynkin diagram is indexed by I and we denote as usual by

- R_+ and $S = \{\alpha_i, i \in S\}$ its subsets of positive and simple roots,
- W its Weyl group with generators $s_i, i \in I$ associated to the simple roots $\alpha_i, i \in S$,
- ℓ the length function on W: for any w in W, $\ell(w)$ is the number of generators s_i in any reduced expression of w,
- Q its root lattice and Q_+ the cone generated by the positive roots,
- P its weight lattice and P_+ the cone of dominant weights generated by the fundamental weights $\omega_i, i \in I$,
- $V(\nu)$ the simple $\mathfrak{g}$-module of highest weight $\nu \in P_+$,
- $\langle \cdot, \cdot \rangle$ the usual Euclidean scalar product on E,
- $\leq$ the dominance order on P such that $\gamma \leq \mu$ if and only if $\mu - \gamma \in Q_+$,
- $$\mathfrak{g} = \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in R_+} \mathfrak{g}_{-\alpha}$$
  the triangular decomposition of $\mathfrak{g}$ where $\mathfrak{h}$ is the Cartan subalgebra.

Now, let $\overline{I}$ be a subset of I and denote by $\overline{R}$ the associated parabolic root system with Weyl group $\overline{W}$, set of positive roots $\overline{R}_+$ and set of dominant weights $\overline{P}_+$. Here we have

$$\overline{R} = R \cap \left(\oplus_{i\in\overline{I}} \mathbb{R}\alpha_i\right) \text{ and } \overline{R}_+ = \overline{R} \cap R_+.$$

Recall that each right coset $\overline{W}w$ admits a unique minimal length element and write U for the set of these minimal length elements. Then each w in W decomposes in a unique way on the form $w = \overline{w}u$ with $u \in U$ and $\overline{w} \in \overline{W}$. We consider the Levi subalgebra

$$\overline{\mathfrak{g}} = \bigoplus_{\alpha\in\overline{R}_+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha\in\overline{R}_+} \mathfrak{g}_{-\alpha}.$$

The algebra $\overline{\mathfrak{g}}$ is only semisimple and has the same weight lattice as $\mathfrak{g}$.
Write $\overline{E} = \oplus_{i\in\overline{I}} \mathbb{R}\omega_i$, the linear real subspace of E generated by the fundamental weights $\omega_i, i \in \overline{I}$ and set $\overline{E}^\diamondsuit = \oplus_{i\in I\setminus\overline{I}} \mathbb{R}\omega_i$. Each vector v in $E = \overline{E} \oplus \overline{E}^\diamondsuit$ decomposes uniquely on the form

$$v = \overline{v} + v^\diamondsuit \tag{1}$$

where $\overline{v} \in \overline{E}$ and $v^\diamondsuit \in \overline{E}^\diamondsuit$. Observe that the elements of $\overline{E}^\diamondsuit$ are fixed under the action of $\overline{W}$ because $\langle \omega_i, \alpha_j\rangle = 0$ for any $j \in \overline{I}$ and $i \in I \setminus \overline{I}$. In fact we have the inclusion

$$\overline{E}^\diamondsuit \subset \{x \in E \mid \langle x, \alpha_i\rangle = 0, i \in \overline{I}\} = \operatorname{span}(\alpha_i, i \in \overline{I})^\perp$$

which is the geometric orthogonal of the linear span of the simple roots $\alpha_i, i \in \overline{I}$.

Lemma 2.1. *For any $\overline{w}$ in $\overline{W}$, we have $\overline{w}(R_+ \setminus \overline{R}_+) = R_+ \setminus \overline{R}_+$, that is the set $R_+ \setminus \overline{R}_+$ is invariant under the action of $\overline{W}$.*

Proof. Consider $\alpha \in R_+ \setminus \overline{R}_+$ and $i \in \overline{I}$. Then $s_i(\alpha)$ should belong to R_+. Indeed α_i is the unique positive root which is changed into a negative root by s_i and $\alpha \neq \alpha_i$ by assumption. Therefore, we can set $s_i(\alpha) = \alpha'$ with α' in R_+. Now, we cannot have α' in $\overline{R}_+$ since otherwise $\alpha = s_i(\alpha')$ would belong to $\overline{R}$ which is stable under the action of $\overline{W}$ and therefore we would have $\alpha \in \overline{R}_+$ hence a contradiction. □

Each root in $R_+ \setminus \overline{R}_+$ is nevertheless not fixed by the action of $\overline{W}$: the action of an element of $\overline{W}$ will permute the roots in $R_+ \setminus \overline{R}_+$.
The simple $\overline{\mathfrak{g}}$-modules are labeled by the weights in the cone $\overline{P}_+ = \{\delta \in P \mid \langle \delta, \alpha_i\rangle \geq 0, i \in \overline{I}\}$. We have

$$\overline{P}_+ = \bigoplus_{i\in\overline{I}} \mathbb{N}\omega_i \oplus \bigoplus_{i\in I\setminus\overline{I}} \mathbb{Z}\omega_i$$

that is, each dominant weight in $\overline{P}_+$ can be written on the form

$$\overline{\lambda} + \gamma \text{ with } \overline{\lambda} \in \bigoplus_{i\in\overline{I}} \mathbb{N}\omega_i \text{ and } \gamma \in \bigoplus_{i\in I\setminus\overline{I}} \mathbb{Z}\omega_i. \tag{2}$$

where γ is fixed under the action of $\overline{W}$. We write $\overline{V}(\overline{\lambda}+\gamma)$ for the simple $\overline{\mathfrak{g}}$-module with highest weight $\overline{\lambda} + \gamma$.

Example 2.2. Assume $\mathfrak{g} = \mathfrak{sp}_{2n}$ is the Lie algebra of type C_n and $\overline{I} = \{1, \dots, n-1\}$ (the longest simple root is α_n). With the usual realization of the root system of type C_n (see [1]), we have $E = \oplus_{i=1}^n \mathbb{R}\varepsilon_i$, $\alpha_i = \varepsilon_i - \varepsilon_{i+1}, i = 1, \dots, n-1$ and $\alpha_n = 2\varepsilon_n$. Moreover $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$. Then $\overline{\mathfrak{g}}$ is isomorphic to $\mathfrak{gl}_n$ and we have

$$\overline{E} = \bigoplus_{i=1}^{n-1} \mathbb{R}\omega_i = \bigoplus_{i=1}^{n-1} \mathbb{R}\varepsilon_i.$$

The dominant weights of $\overline{\mathfrak{g}}$ are the generalized partitions $\kappa = (\kappa_1 \geq \cdots \geq \kappa_n) \in \mathbb{Z}^n$ and we can write write

$$\kappa = \overline{\lambda} + \gamma$$

with $\overline{\lambda} = (\kappa_1 - \kappa_n, \ldots, \kappa_{n-1} - \kappa_n, 0)$ and $\gamma = \kappa_n \omega_n$.

Consider q and p two formal indeterminates and introduce the p,q-Kostant partitions $\mathrm{P}_{p,q}$ defined from the expansion of the formal series

$$\prod_{\alpha \in R_+ \setminus \overline{R}_+} \frac{1}{1 - pe^{\alpha}} \prod_{\alpha \in \overline{R}_+} \frac{1}{1 - qe^{\alpha}} = \sum_{\beta \in Q_+} \mathrm{P}_{p,q}(\beta) e^{\beta}. \tag{3}$$

When $p = q$, this is the ordinary q-partition function (see [9]). Given two weights ν and μ in P with ν dominant, set

$$K_{\nu,\mu}(p,q) = \sum_{w \in W} \varepsilon(w) \mathrm{P}_{p,q}(w(\nu + \rho) - \mu - \rho) \tag{4}$$

where $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$ and $\varepsilon(w) = (-1)^{\ell(w)}$. When $p = q = 1$, the integer $K_{\nu,\mu} = K_{\nu,\mu}(1,1)$ gives the dimension of the weight space

$$V(\nu)_\mu = \{v \in V(\nu) \mid h(v) = \mu(v)v, h \in \mathfrak{h}\}.$$

Now when $p = q$, the polynomial $K_{\nu,\mu}(q,q)$ is the ordinary Lusztig q-analogue of $\dim(V(\nu)_\mu)$.

Remark 2.3. We have $w(\lambda + \rho) - \mu - \rho = w(\lambda + \rho) - (\lambda + \rho) + (\lambda - \mu)$ and $(\lambda + \rho) - w(\lambda + \rho)$ belongs to Q_+. Also for any weight $\gamma \in P$, we have $\mathrm{P}_{p,q}(\gamma) \neq 0$ only if $\gamma \in Q_+$. Therefore, the polynomial $K_{\nu,\mu}(p,q)$ is nonzero only if $\mu \leq \lambda$.

Let us now define the partition function $\hat{\mathrm{P}}_p$ from the expansion of the formal series

$$\prod_{\alpha \in R_+ \setminus \overline{R}_+} \frac{1}{1 - pe^{\alpha}} = \sum_{\eta \in \hat{Q}_+} \hat{\mathrm{P}}_p(\eta) e^{\eta}$$

where $\hat{Q}_+ = \sum_{\alpha \in R_+ \setminus \overline{R}_+} \mathbb{N}\alpha$. Since we have $\overline{w}(R_+ \setminus \overline{R}_+) = R_+ \setminus \overline{R}_+$ for any $\overline{w} \in \overline{W}$, the cone $\hat{Q}_+$ is also stable under the action of $\overline{W}$. Moreover, for any $\overline{w} \in \overline{W}$ and any $\eta \in \hat{Q}_+$, we have $\hat{\mathrm{P}}_p(\overline{w}(\eta)) = \hat{\mathrm{P}}_p(\eta)$. This is because $\overline{w}$ permutes the roots in $R_+ \setminus \overline{R}_+$. Given a dominant weight $\nu \in P_+$ and a parabolic dominant weight $\overline{\lambda} + \gamma \in \overline{P}_+$ as in (2) define the polynomial

$$b_{\nu, \overline{\lambda} + \gamma}(p) = \sum_{w \in W} \varepsilon(w) \hat{\mathrm{P}}_p(w(\nu + \rho) - \overline{\lambda} - \gamma - \rho). \tag{5}$$

Then, it is classical to deduce from the Weyl character formula (see for example [6]) that $b_{\nu, \overline{\lambda} + \gamma}(1)$ gives the multiplicity of $\overline{V}(\overline{\lambda} + \gamma)$ in the restriction of $V(\nu)$ from $\mathfrak{g}$ to $\overline{\mathfrak{g}}$. Observe that in general, the polynomial $b_{\nu, \overline{\lambda} + \gamma}(p)$ does not belong to $\mathbb{N}[p]$. We nevertheless mention the following conjecture which, as far as we are aware, is only partially proved (in particular by Broer [2] in type A). We also refer to [21], [24] and the references therein for more details and results on this problem.

Conjecture 2.4. *The polynomial $b_{\nu, \overline{\lambda} + \gamma}(p)$ has nonnegative integer coefficients at least when $\overline{\lambda} + \gamma \in P_+$, that is when $\overline{\lambda} + \gamma$ is a dominant weight for $\mathfrak{g}$ (and not only for $\overline{\mathfrak{g}}$).*

Example 2.5.

(1) Assume $\mathfrak{g} = \mathfrak{sp}_{10}$ with $\overline{I} = \{1, 2, 4, 5\}$ so that $\overline{\mathfrak{g}} = \mathfrak{sp}_4 \oplus \mathfrak{gl}_1 \oplus \mathfrak{gl}_2$. Put $\nu = (4, 3, 3, 2, 1)$ and $\overline{\lambda} + \gamma = (1, 0, 0, 0, 0)$. Then

$$b_{\nu,\overline{\lambda}+\gamma}(p) = 6p^{11} + 16p^{10} + 21p^9 + 14p^8 + 5p^7$$

where we have an alternated sum of 73 nonzero polynomials in (5).

(2) Assume $\mathfrak{g} = \mathfrak{gl}_3$ with $I = \{1, 2\}$ and assume $\overline{I} = \{1\}$ so that $\overline{\mathfrak{g}} = \mathfrak{gl}_2 \oplus \mathfrak{gl}_1$ (here we use $\mathfrak{gl}_3$ instead of $\mathfrak{sl}_3$ for simplicity). Put $\nu = (6, 3, 1)$ and $\overline{\lambda} + \gamma = (2, 4, 4)$. Then

$$b_{\nu,\overline{\lambda}+\gamma}(p) = -p^3$$

does not have nonnegative coefficients. This shows that the assumption $\overline{\lambda} + \gamma$ is a dominant weight for $\mathfrak{g}$ (and not only for $\overline{\mathfrak{g}}$) is crucial in the previous conjecture.

Write $\overline{\mathrm{P}}_q$ for the usual partition function associated to the root system $\overline{R}$ defined by

$$\prod_{\alpha \in \overline{R}_+} \frac{1}{1 - qe^{\alpha}} = \sum_{\delta \in \overline{Q}_+} \overline{\mathrm{P}}_q(\delta) e^{\delta}.$$

Given two dominant weights $\overline{\lambda} + \gamma$ and $\overline{\mu} + \gamma'$ in $\overline{P}_+$ written as in (2), we can consider the Lusztig q-analogue for the subalgebra $\overline{\mathfrak{g}}$

$$\overline{K}_{\overline{\lambda}+\gamma,\overline{\mu}+\gamma'}(q) = \sum_{\overline{w} \in \overline{W}} \varepsilon(\overline{w}) \overline{\mathrm{P}}_q(\overline{w}(\overline{\lambda} + \gamma + \overline{\rho}) - \overline{\mu} - \gamma' - \overline{\rho}).$$

Since $\overline{w}(\gamma) = \gamma$ for any $\overline{w} \in \overline{W}$, we have $\overline{w}(\overline{\lambda} + \gamma + \overline{\rho}) - \overline{\mu} - \gamma' - \overline{\rho} = \overline{w}(\overline{\lambda} + \overline{\rho}) - \overline{\mu} - \overline{\rho} + \gamma - \gamma'$. Also it follows from the previous definition that $\overline{w}(\overline{\lambda} + \overline{\rho}) - \overline{\mu} - \overline{\rho}$ belongs to $\overline{E}$ whereas $\gamma - \gamma'$ belongs to $\overline{E}^{\diamondsuit}$. Moreover, we have $\overline{\mathrm{P}}_q(\beta) \neq 0$ only if β belongs to $\overline{E}$. Thus, we can conclude that $\overline{K}_{\overline{\lambda}+\gamma,\overline{\mu}+\gamma'}(q)$ is nonzero only when $\gamma = \gamma'$ in which case we moreover have

$$\overline{K}_{\overline{\lambda}+\gamma,\overline{\mu}+\gamma}(q) = \overline{K}_{\overline{\lambda},\overline{\mu}}(q). \tag{6}$$

We can also enlarge the definition of the polynomials $\overline{K}_{\overline{\lambda},\overline{\mu}}(q)$ by replacing the dominant weight $\overline{\lambda} \in \bigoplus_{i \in \overline{I}} \mathbb{N}\varpi_i$ by any weight $\overline{\kappa} \in \bigoplus_{i \in \overline{I}} \mathbb{Z}\varpi_i$. Then we get $\overline{K}_{\overline{\kappa},\overline{\mu}}(q) = 0$ or there exists $\overline{w}$ in $\overline{W}$ and $\overline{\lambda} \in \overline{P}_+$ such that

$$\overline{w}^{-1}(\overline{\lambda} + \overline{\rho}) - \overline{\rho} = \overline{\kappa} \text{ and } \overline{K}_{\overline{\kappa},\overline{\mu}}(q) = \varepsilon(\overline{w}) \overline{K}_{\overline{\lambda},\overline{\mu}}(q).$$

Finally, by elementary properties of formal series, we must have for any $\beta \in Q_+$

$$\mathrm{P}_{p,q}(\beta) = \sum_{\gamma \in \hat{Q}_+, \delta \in \overline{Q}_+ \mid \gamma + \delta = \beta} \hat{\mathrm{P}}_p(\gamma) \overline{\mathrm{P}}_q(\delta) = \sum_{\gamma \in \hat{Q}_+} \hat{\mathrm{P}}_p(\gamma) \overline{\mathrm{P}}_q(\beta - \gamma). \tag{7}$$

We also recall the Weyl character formula. For each dominant weight ν in P_+, the character of $V(\nu)$ is the polynomial in $\mathbb{Z}^W[P]$ such that

$$s_\nu = \frac{\sum_{w \in W} \varepsilon(w) e^{w(\nu+\rho)-\rho}}{\prod_{\alpha \in R_+} (1 - e^{-\alpha})}.$$

In particular for $\nu = 0$, we get the Weyl denominator formula

$$\prod_{\alpha \in R_+} (1 - e^{-\alpha}) = \sum_{w \in W} \varepsilon(w) e^{w(\rho)-\rho}.$$

We shall also use the notation $a_\gamma = \sum_{w \in W} \varepsilon(w) e^{w(\gamma)}$ for any $\gamma \in P$. Then $s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho}$.

3. Decomposition of the double deformation

In this section, we establish a decomposition theorem for our double deformations of weight multiplicities. When $p=q=1$, it just means that one can obtain the restriction of any representation of $\mathfrak{g}$ to its Cartan subalgebra $\mathfrak{h}$ by performing first its restriction to $\overline{\mathfrak{g}}$ and next the restriction from $\overline{\mathfrak{g}}$ to $\mathfrak{h}$. When we have only $p=1$, we interpret the polynomials $K_{\nu,\mu}(1,q)$ in terms of the parabolic Brylinski filtration of $V(\nu)$ obtained from $\overline{\mathfrak{g}}$.

3.1. **The general case.**

Theorem 3.1. *For any dominant weight ν in P_+ and any weight μ in P we have the decomposition*

$$K_{\nu,\mu}(p,q)=\sum_{\overline{\lambda}\in\overline{P}_+\cap\overline{E}} b_{\nu,\overline{\lambda}+\mu^{\diamondsuit}}(p)\overline{K}_{\overline{\lambda},\overline{\mu}}(q)=\sum_{\overline{\lambda}\in\overline{P}_+\cap\overline{E}} b_{\nu,\overline{\lambda}+\mu^{\diamondsuit}}(p)\overline{K}_{\overline{\lambda}+\mu^{\diamondsuit},\mu}(q)$$

where $\overline{\lambda}+\mu^{\diamondsuit}$ belongs to $\overline{P}_+$.

Proof. By definition of the polynomial $K_{\nu,\mu}(p,q)$ and by using (7), we obtain

$$K_{\nu,\mu}(p,q)=\sum_{u\in U}\varepsilon(u)\sum_{\gamma\in\hat{Q}_+}\hat{\mathrm{P}}_p(\gamma)\sum_{\overline{w}\in\overline{W}}\varepsilon(\overline{w})\overline{\mathrm{P}}_q(\overline{w}u(\nu+\rho)-\mu-\rho-\gamma)$$

where we have decomposed each w in W on the form $w=\overline{w}u$ with $u\in U$ and $\overline{w}\in\overline{W}$. Since $\hat{Q}_+$ is stable under the action of $\overline{W}$ and $\hat{\mathrm{P}}_p(\overline{w}^{-1}(\gamma))=\hat{\mathrm{P}}_p(\gamma)$ for any $\overline{w}\in\overline{W}$ and $\gamma\in\hat{Q}_+$, we also have

$$K_{\nu,\mu}(p,q)=\sum_{u\in U}\varepsilon(u)\sum_{\gamma'\in\hat{Q}_+}\hat{\mathrm{P}}_p(\gamma')\sum_{\overline{w}\in\overline{W}}\varepsilon(\overline{w})\overline{\mathrm{P}}_q(\overline{w}u(\nu+\rho)-\mu-\rho-\overline{w}(\gamma'))$$

by setting $\gamma'=\overline{w}^{-1}(\gamma)$.
We can now apply decompositions of type (1) to get $\rho=\overline{\rho}+\rho^{\diamondsuit}$ and $\mu=\overline{\mu}+\mu^{\diamondsuit}$. By using that $\overline{w}(\rho^{\diamondsuit})=\rho^{\diamondsuit}$ and $\overline{w}(\mu^{\diamondsuit})=\mu^{\diamondsuit}$, we obtain

$$\begin{aligned}\overline{w}\left(u(\nu+\rho)\right)-\mu-\rho-\overline{w}(\gamma')&=\overline{w}[u(\nu+\rho)-\mu^{\diamondsuit}-\rho^{\diamondsuit}-\gamma']-\overline{\rho}-\overline{\mu}=\\&\overline{w}[(u(\nu+\rho)-\mu^{\diamondsuit}-\rho^{\diamondsuit}-\gamma'-\overline{\rho})+\overline{\rho}]-\overline{\rho}-\overline{\mu}=\\&\overline{w}[(u(\nu+\rho)-\mu^{\diamondsuit}-\rho-\gamma')+\overline{\rho}]-\overline{\rho}-\overline{\mu}.\end{aligned}$$

This gives

$$\overline{\mathrm{P}}_q(\overline{w}u(\nu+\rho)-\mu-\rho-\overline{w}(\gamma'))=\overline{\mathrm{P}}_q\left(\overline{w}[(u(\nu+\rho)-\rho-\mu^{\diamondsuit}-\gamma')+\overline{\rho}]-\overline{\rho}-\overline{\mu}\right).$$

Therefore, we have

$$\sum_{\gamma'\in\hat{Q}_+}\hat{\mathrm{P}}_p(\gamma')\sum_{\overline{w}\in\overline{W}}\varepsilon(\overline{w})\overline{\mathrm{P}}_q(\overline{w}u(\nu+\rho)-\mu-\rho-\overline{w}(\gamma'))=\sum_{\gamma'\in\hat{Q}_+}\hat{\mathrm{P}}_p(\gamma')\overline{K}_{u(\nu+\rho)-\rho-\mu^{\diamondsuit}-\gamma',\overline{\mu}}(q).$$

This yields

$$K_{\nu,\mu}(p,q)=\sum_{u\in U}\varepsilon(u)\sum_{\gamma'\in\hat{Q}_+}\hat{\mathrm{P}}_p(\gamma')\overline{K}_{u(\nu+\rho)-\rho-\mu^{\diamondsuit}-\gamma',\overline{\mu}}(q).$$

For each nonzero polynomial $\overline{K}_{u(\nu+\rho)-\rho-\mu^{\diamondsuit}-\gamma',\overline{\mu}}(q)$ there exists $\overline{w}$ in $\overline{W}$ and $\overline{\lambda}\in\overline{P}_+$ such that

$$\overline{w}^{-1}(\overline{\lambda}+\overline{\rho})-\overline{\rho}=u(\nu+\rho)-\rho-\mu^{\diamondsuit}-\gamma'\text{ and }\overline{K}_{u(\nu+\rho)-\rho-\mu^{\diamondsuit}-\gamma',\overline{\mu}}(q)=\varepsilon(\overline{w})\overline{K}_{\overline{\lambda},\overline{\mu}}(q).$$

We then have

$$\gamma'=u(\nu+\rho)-\rho-\mu^{\diamondsuit}-\overline{w}^{-1}(\overline{\lambda}+\overline{\rho})+\overline{\rho}=u(\nu+\rho)-\rho^{\diamondsuit}-\mu^{\diamondsuit}-\overline{w}^{-1}(\overline{\lambda}+\overline{\rho}).$$

By gathering all the contributions corresponding to the same dominant weight $\overline{\lambda}$, we thus obtain

$$K_{\nu,\mu}(p,q)=\sum_{\overline{\lambda}\in\overline{P}_+}\left(\sum_{u\in U}\varepsilon(u)\sum_{\overline{w}\in\overline{W}}\varepsilon(\overline{w})\hat{\mathrm{P}}_p\left(u(\nu+\rho)-\rho^{\diamondsuit}-\mu^{\diamondsuit}-\overline{w}^{-1}(\overline{\lambda}+\overline{\rho})\right)\right)\overline{K}_{\overline{\lambda},\overline{\mu}}(q).$$

Recall that for any $\eta\in\hat{Q}_+$ and any $\overline{w}\in\overline{W}$, we have $\hat{\mathrm{P}}_p(\overline{w}(\eta))=\hat{\mathrm{P}}_p(\eta)$. We also have $\overline{w}(\rho^{\diamondsuit})=\rho^{\diamondsuit}$ and $\overline{w}(\mu^{\diamondsuit})=\mu^{\diamondsuit}$ and thus

$$\begin{aligned}\hat{\mathrm{P}}_p\left(u(\nu+\rho)-\rho^{\diamondsuit}-\mu^{\diamondsuit}-\overline{w}^{-1}(\overline{\lambda}+\overline{\rho})\right)&=\hat{\mathrm{P}}_p\left(\overline{w}u(\nu+\rho)-\rho^{\diamondsuit}-\mu^{\diamondsuit}-\overline{\lambda}-\overline{\rho})\right)\\&=\hat{\mathrm{P}}_p\left(\overline{w}u(\nu+\rho)-\mu^{\diamondsuit}-\overline{\lambda}-\rho)\right).\end{aligned}$$

This permits to write

$$K_{\nu,\mu}(p,q)=\sum_{\overline{\lambda}\in\overline{P}_+}\left(\sum_{u\in U}\sum_{\overline{w}\in\overline{W}}\varepsilon(\overline{w}u)\hat{\mathrm{P}}_p\left(\overline{w}u(\nu+\rho)-\mu^{\diamondsuit}-\overline{\lambda}-\rho)\right)\right)\overline{K}_{\overline{\lambda},\overline{\mu}}(q)=$$
$$\sum_{\overline{\lambda}\in\overline{P}_+}\left(\sum_{w\in W}\varepsilon(w)\hat{\mathrm{P}}_p\left(w(\nu+\rho)-\mu^{\diamondsuit}-\overline{\lambda}-\rho)\right)\right)\overline{K}_{\overline{\lambda},\overline{\mu}}(q)=\sum_{\overline{\lambda}\in\overline{P}_+\cap\overline{E}}b_{\nu,\overline{\lambda}+\mu^{\diamondsuit}}(p)\overline{K}_{\overline{\lambda},\overline{\mu}}(q)\tag{8}$$

as claimed. Here the restriction of the last sum over $\overline{P}_+\cap\overline{E}$ comes from the fact that $\overline{K}_{\overline{\lambda},\overline{\mu}}(q)\neq 0$ only if $\overline{\lambda}-\overline{\mu}$ belongs to $\overline{E}$ and we have here $\overline{\mu}\in\overline{E}$. It is straightforward to check that $\overline{\lambda}+\mu^{\diamondsuit}$ belongs to $\overline{P}_+$ since $\mu^{\diamondsuit}$ belongs to $\overline{E}^{\diamondsuit}$ and thus $\langle\overline{\lambda}+\mu^{\diamondsuit},\alpha_i\rangle=\langle\overline{\lambda},\alpha_i\rangle\geq 0$ for any $i\in\overline{I}$. The last equality of the theorem follows immediately from Equality (6) since we have $\mu=\overline{\mu}+\mu^{\diamondsuit}$. □

3.2. **The case $p=1$ and the parabolic Brylinski-Kostant filtration.** When $p=1$, we obtain immediately from Theorem 3.1 the decomposition

$$K_{\nu,\mu}(1,q)=\sum_{\overline{\lambda}\in\overline{P}_+\cap\overline{E}}b_{\nu,\overline{\lambda}+\mu^{\diamondsuit}}\overline{K}_{\overline{\lambda}+\mu^{\diamondsuit},\mu}(q)\tag{9}$$

where the nonnegative integer $b_{\nu,\overline{\lambda}+\mu^{\diamondsuit}}$ is the branching coefficient giving the multiplicity of the $\overline{\mathfrak{g}}$-module $\overline{V}(\overline{\lambda}+\mu^{\diamondsuit})$ in the restriction of $V(\nu)$ from $\mathfrak{g}$ to $\overline{\mathfrak{g}}$. In particular, when $\overline{\mu}$ belongs to $\overline{P}_+$, we get that $K_{\nu,\mu}(1,q)$ belongs to $\mathbb{N}[q]$ since each polynomial $\overline{K}_{\overline{\lambda}+\mu^{\diamondsuit},\mu}(q)=\overline{K}_{\overline{\lambda},\overline{\mu}}(q)$ does. It is then possible to give an algebraic interpretation of the polynomial $K_{\nu,\mu}(1,q)$ in terms of the parabolic Brylinski-Kostant filtration corresponding to $\overline{\mathfrak{g}}$. To do so, denote by $e_i, i\in I$ the raising Chevalley generators[1] of the Lie algebra $\mathfrak{g}$ and set

$$\overline{e}=\sum_{i\in\overline{I}}e_i.$$

Then the element $\overline{e}$ is a principal nilpotent for the Lie algebra $\overline{\mathfrak{g}}$.
Now assume that $\nu\in P_+$ is fixed. For any nonnegative integer k, consider the increasing sequence (for the inclusion) of linear subspaces

$$\overline{J}^{(k)}(\nu)_\mu:=\{v\in V(\nu)_\mu\mid \overline{e}^{(k)}(v)=0\}.$$

[1]We assume here that we choose a set of Chevalley generators compatible with the triangular decomposition of $\mathfrak{g}$ considered.

The space $\overline{J}^{(k)}(\nu)$ is thus the intersection of the weight space $V(\nu)_\mu$ with the kernel of the action of the element $\overline{e}^k$ on the $\mathfrak{g}$-module $V(\nu)$. We can define the jump polynomial $\mathcal{K}_{\nu,\mu}(q)$ by setting

$$\mathcal{K}_{\nu,\mu}(q)=\sum_{k\geq 0}\left(\dim\left(\overline{J}^{(k+1)}(\nu)_\mu\right)-\dim\left(\overline{J}^{(k)}(\nu)_\mu\right)\right)q^k.$$

Theorem 3.2. *For any $\nu\in P_+$ and any $\mu=\overline{\mu}+\mu^\diamondsuit\in\overline{P}_+$, we have*

$$\mathcal{K}_{\nu,\mu}(q)=K_{\nu,\mu}(1,q).$$

Proof. The $\mathfrak{g}$-module $V(\nu)$ decomposes as a direct sum of irreducible $\overline{\mathfrak{g}}$-modules. Let $\overline{M}$ be one of these $\overline{\mathfrak{g}}$-modules. Assume that $\overline{M}$ has highest weight $\overline{\lambda}+\gamma\in\overline{P}_+$ (decomposed as in (2)) and $\overline{M}\cap V(\nu)_\mu\neq\{0\}$. Then, the difference $\overline{\lambda}+\gamma-\mu$ is a sum of simple roots $\alpha_i, i\in\overline{I}$ and therefore belongs to $\overline{E}$. It follows that $\overline{\lambda}+\gamma$ and μ have the same component on $\overline{E}^\diamondsuit$, that is $\gamma=\mu^\diamondsuit$. Recall that the decomposition of $V(\nu)$ into its weight spaces refines the decomposition of $V(\nu)$ into its irreducible $\overline{\mathfrak{g}}$-components. By considering only the irreducible $\overline{\mathfrak{g}}$-modules with highest weight of the form $\overline{\lambda}+\mu^\diamondsuit,\overline{\lambda}\in\overline{P}_+\cap\overline{E}$ and their multiplicities $b_{\nu,\overline{\lambda}+\mu^\diamondsuit}$ in the decomposition of $V(\nu)$, we therefore get the isomorphism

$$V(\nu)_\mu\simeq\bigoplus_{\overline{\lambda}\in\overline{P}_+\cap\overline{E}}\left(\overline{V}(\overline{\lambda}+\mu^\diamondsuit)\cap V(\nu)_\mu\right)^{\oplus b_{\nu,\overline{\lambda}+\mu^\diamondsuit}}=\bigoplus_{\overline{\lambda}\in\overline{P}_+\cap\overline{E}}\overline{V}(\overline{\lambda}+\mu^\diamondsuit)_\mu^{\oplus b_{\nu,\overline{\lambda}+\mu^\diamondsuit}}$$

where $\overline{V}(\overline{\lambda}+\mu^\diamondsuit)_\mu$ is the weight space associated to μ in $\overline{V}(\overline{\lambda}+\mu^\diamondsuit)$. Since each $\overline{\mathfrak{g}}$-module is stable under the action of $\overline{e}$, this also gives for any $k\geq 0$

$$\overline{J}^{(k)}(\nu)_\mu\simeq\bigoplus_{\overline{\lambda}\in\overline{P}_+\cap\overline{E}}\left(\overline{V}(\overline{\lambda}+\mu^\diamondsuit)_\mu\cap\overline{J}^{(k)}(\nu)_\mu\right)^{\oplus b_{\nu,\overline{\lambda}+\mu^\diamondsuit}}$$

and

$$\dim\left(\overline{J}^{(k)}(\nu)_\mu\right)=\sum_{\overline{\lambda}\in\overline{P}_+\cap\overline{E}}b_{\nu,\overline{\lambda}+\mu^\diamondsuit}\dim\left(\overline{V}(\overline{\lambda}+\mu^\diamondsuit)_\mu\cap\overline{J}^{(k)}(\nu)_\mu\right).$$

We so obtain

$$\begin{aligned}&\mathcal{K}_{\nu,\mu}(q)=\\&\sum_{k\geq 0}\sum_{\overline{\lambda}\in\overline{P}_+\cap\overline{E}}b_{\nu,\overline{\lambda}+\mu^\diamondsuit}\left(\dim\left(\overline{V}(\overline{\lambda}+\mu^\diamondsuit)_\mu\cap\overline{J}^{(k+1)}(\nu)_\mu\right)-\dim\left(\overline{V}(\overline{\lambda}+\mu^\diamondsuit)_\mu\cap\overline{J}^{(k)}(\nu)_\mu\right)\right)q^k\\&=\sum_{\overline{\lambda}\in\overline{P}_+\cap\overline{E}}b_{\nu,\overline{\lambda}+\mu^\diamondsuit}\sum_{k\geq 0}\left(\dim\left(\overline{V}(\overline{\lambda}+\mu^\diamondsuit)_\mu\cap\overline{J}^{(k+1)}(\nu)_\mu\right)-\dim\left(\overline{V}(\overline{\lambda}+\mu^\diamondsuit)_\mu\cap\overline{J}^{(k)}(\nu)_\mu\right)\right)q^k.\end{aligned}$$

For any $\overline{\lambda}\in\overline{P}_+\cap\overline{E}$, Brylinski's theorem (see [3]) equates the jump polynomial in $\overline{V}(\overline{\lambda}+\mu^\diamondsuit)$ for the weight μ with the parabolic Lusztig's q-analogue $\overline{K}_{\overline{\lambda}+\mu^\diamondsuit,\mu}(q)$ so that

$$\overline{K}_{\overline{\lambda}+\mu^\diamondsuit,\mu}(q)=\sum_{k\geq 0}\left(\dim\left(\overline{V}(\overline{\lambda}+\mu^\diamondsuit)_\mu\cap\overline{J}^{(k+1)}(\nu)_\mu\right)-\dim\left(\overline{V}(\overline{\lambda}+\mu^\diamondsuit)_\mu\cap\overline{J}^{(k)}(\nu)_\mu\right)\right)q^k.$$

Finally, we have

$$\mathcal{K}_{\nu,\mu}(q)=\sum_{\overline{\lambda}\in\overline{P}_+\cap\overline{E}}b_{\nu,\overline{\lambda}+\mu^\diamondsuit}\overline{K}_{\overline{\lambda}+\mu^\diamondsuit,\mu}(q)=K_{\nu,\mu}(1,q)$$

by (9). □

3.3. **Other positivity results.** We summarize in this paragraph what can be immediately claimed about the nonnegativity of the coefficients of the polynomials $K_{\nu,\mu}(p,q)$ (whose coefficients belong a priori to $\mathbb{Z}$). We get the following direct corollary of our Theorem 3.1.

Corollary 3.3. *Consider ν a dominant weight for $\mathfrak{g}$. We have for any weight $\mu \in P$*

$$\begin{aligned}
K_{\nu,\mu}(1,q) &= \sum_{\overline{\lambda}\in \overline{P}_+} b_{\nu,\overline{\lambda}+\mu^{\diamondsuit}}(1)\overline{K}_{\overline{\lambda},\overline{\mu}}(q),\\
K_{\nu,\mu}(0,q) &= \overline{K}_{\overline{\nu},\overline{\mu}}(q),\\
K_{\nu,\mu}(p,0) &= b_{\nu,\mu}(p),\\
K_{\nu,\mu}(p,p) &= K_{\nu,\mu}(p).
\end{aligned}$$

Moreover,

- *the polynomial $K_{\nu,\mu}(1,q)$ has nonnegative integer coefficients when $\mu \in \overline{P}_+$,*
- *the polynomial $K_{\nu,\mu}(q,q)$ has nonnegative integer coefficients when $\mu \in P_+$,*
- *the polynomial $K_{\nu,\mu}(p,0)$ has nonnegative in the cases where Conjecture 2.4 holds.*

Proof. The case $p=1$ has already been studied in § 3.2.
Now when $p=0$, we have $\hat{\mathrm{P}}_p(\gamma)=\delta_{1,\gamma}$ for any $\gamma \in \hat{Q}_+$. Therefore for any $\overline{\lambda}\in\overline{P}_+$, we get $b_{\nu,\overline{\lambda}+\mu^{\diamondsuit}}(0)\neq 0$ if and only if $\nu=\overline{\lambda}+\mu^{\diamondsuit}$. This means that $\overline{\lambda}=\overline{\nu}$ and $\mu^{\diamondsuit}=\nu^{\diamondsuit}$. In this case we get the equality $b_{\nu,\overline{\nu}+\nu^{\diamondsuit}}(0)=b_{\nu,\nu}(0)=1$. This therefore gives

$$K_{\nu,\mu}(0,q)=\overline{K}_{\overline{\nu},\overline{\mu}}(q).$$

When $q=0$, we have $\overline{K}_{\overline{\lambda},\overline{\mu}}(0)\neq 0$ only when $\overline{\lambda}=\overline{\mu}$ and then $\overline{K}_{\overline{\lambda},\overline{\mu}}(0)=1$. Thus by Theorem 3.1, we get

$$K_{\nu,\mu}(p,0)=b_{\nu,\overline{\mu}+\mu^{\diamondsuit}}(p)=b_{\nu,\mu}(p)$$

which has nonnegative coefficients in the cases where Conjecture 2.4 holds.
The last identity $K_{\nu,\mu}(q,q)=K_{\nu,\mu}(q)$ follows directly from the definition of our polynomials $K_{\nu,\mu}(p,q)$. □

Remark 3.4.

(1) When $p=1$, $\mathfrak{g}$ is one of the classical Lie algebras of type B_n, C_n, D_n and $\overline{\mathfrak{g}}$ its Levi subalgebra of type A_{n-1} obtained by removing the node n in the Dynkin diagram, the positivity of the polynomials $K_{\nu,\mu}(1,q)$ was conjectured by Lee in [17]. This conjecture was then established in [4] in these particular cases (essentially at the same time as this article appeared).

(2) Observe that when λ and μ belong to P_+, the polynomials $K_{\nu,\mu}(p,q)$ do not have nonnegative coefficients in general. For example, when $\mathfrak{g}$ is of type C_2 and $\overline{\mathfrak{g}}$ of type A_1; with $\nu=(4,2)$ and $\mu=(0,0)$, we get

$$K_{\nu,\mu}(p,q)=p^3q^4+p^3q^3+2p^3q^2+p^3q+p^3-q^4.$$

4. The double deformation $K_{\nu,\mu}(p+1,q+1)$

Our goal in this section is to prove that the polynomials $K_{\nu,\mu}(p+1,q+1)$ have nonnegative coefficients for any dominant weight λ and any weight μ. This follows by adapting arguments due to Panyushev in [22] for the classical one-parameter Lusztig q-analogue. Set

$$\Delta=\frac{\prod_{\alpha\in R_+}(1-e^{-\alpha})}{\prod_{\alpha\in R_+\setminus\overline{R}_+}(1-(p+1)e^{-\alpha})\prod_{\alpha\in\overline{R}_+}(1-(q+1)e^{-\alpha})}=$$

$$\frac{e^{-\rho}a_{\rho}}{\prod_{\alpha\in R_{+}\setminus\overline{R}_{+}}(1-(p+1)e^{-\alpha})\prod_{\alpha\in\overline{R}_{+}}(1-(q+1)e^{-\alpha})}$$

Theorem 4.1.

(1) *We have* $\Delta=\sum_{\mu\in Q_{+}}K_{0,-\mu}(p+1,q+1)e^{-\mu}$ *where each polynomial* $K_{0,-\mu}(p+1,q+1)$ *belongs to* $\mathbb{N}[p,q]$.

(2) *For* $\nu\in P_{+}$ *and* $\mu\in P$, *the polynomial* $K_{\nu,\mu}(p+1,q+1)$, *regarded as a polynomial in* $\mathbb{Z}[p,q]$, *has nonnegative coefficients. Moreover, we have*

$$\sum_{\mu\in P}K_{\nu,\mu}(p+1,q+1)e^{\mu}=\Delta s_{\nu}$$

and the decomposition

$$K_{\nu,\mu}(p+1,q+1)=\sum_{\beta\in Q_{+}}K_{\nu,\mu+\beta}K_{0,-\beta}(p+1,q+1)\in\mathbb{N}[p,q]. \tag{10}$$

Proof. We resume the notation of Section 2. First, there should exist some polynomials $m_{\mu}(p+1,q+1)\in\mathbb{Z}[p+1,q+1],\mu\in Q_{+}$ such that

$$\Delta=\frac{\prod_{\alpha\in R_{+}\setminus\overline{R}_{+}}(1-e^{-\alpha})}{\prod_{\alpha\in R_{+}\setminus\overline{R}_{+}}(1-(p+1)e^{-\alpha})}\times\frac{\prod_{\alpha\in\overline{R}_{+}}(1-e^{-\alpha})}{\prod_{\alpha\in\overline{R}_{+}}(1-(q+1)e^{-\alpha})}=\sum_{\mu\in Q_{+}}m_{\mu}(p+1,q+1)e^{-\mu}. \tag{11}$$

Recall that we have

$$\prod_{\alpha\in R_{+}}\left(1-e^{-\alpha}\right)=\sum_{w\in W}\varepsilon(w)e^{w(\rho)-\rho}\text{ and}$$

$$\prod_{\alpha\in R_{+}\setminus\overline{R}_{+}}\frac{1}{1-(p+1)e^{-\alpha}}\prod_{\alpha\in\overline{R}_{+}}\frac{1}{1-(q+1)e^{-\alpha}}=\sum_{\beta\in Q_{+}}\mathrm{P}_{p+1,q+1}(\beta)e^{-\beta}.$$

This gives

$$\Delta=\sum_{\beta\in Q_{+}}\sum_{w\in W}\varepsilon(w)\mathrm{P}_{p+1,q+1}(\beta)e^{w(\rho)-\rho-\beta}=$$

$$\sum_{\mu\in Q_{+}}\sum_{w\in W}\varepsilon(w)\mathrm{P}_{p+1,q+1}(w(\rho)-\rho+\mu)e^{-\mu}=\sum_{\mu\in Q_{+}}K_{0,-\mu}(p+1,q+1)e^{-\mu}$$

by setting $\mu=\beta+\rho-w(\rho)$ which permits to conclude that $m_{\mu}(p+1,q+1)=K_{0,-\mu}(p+1,q+1)$ for any $\mu\in Q_{+}$. We also have for any positive root α

$$\frac{1-e^{-\alpha}}{1-(p+1)e^{-\alpha}}=1+\sum_{k\geq1}p(p+1)^{k-1}e^{-k\alpha} \tag{12}$$

and a similar identity by replacing p by q. Therefore the formal expansion of Δ should have nonnegative integer coefficients as claimed.

Now introduce the generating series

$$u_{\nu}:=\sum_{\mu\in P}K_{\nu,\mu}(p+1,q+1)e^{\mu}.$$

We have

$$u_{\nu}=\sum_{\mu\in P}\sum_{w\in W}\varepsilon(w)\mathrm{P}_{p+1,q+1}(w(\nu+\rho)-\mu-\rho)e^{\mu}=\sum_{w\in W}\varepsilon(w)\sum_{\gamma\in P}\mathrm{P}_{p+1,q+1}(\gamma)e^{w(\nu+\rho)-\gamma-\rho}=$$

$$\sum_{w\in W}\varepsilon(w)\left(\sum_{\gamma\in P}\mathrm{P}_{p+1,q+1}(\gamma)e^{-\gamma}\right)e^{w(\nu+\rho)-\rho}=\frac{e^{-\rho}\sum_{w\in W}\varepsilon(w)e^{w(\nu+\rho)}}{\prod_{\alpha\in R_+\setminus\overline{R}_+}(1-(p+1)e^{-\alpha})\prod_{\alpha\in\overline{R}_+}(1-(q+1)e^{-\alpha})}$$
$$=\Delta\frac{a_{\nu+\rho}}{a_\rho}=\Delta s_\nu$$

where we set $\gamma=w(\nu+\rho)-\mu-\rho$ in the second equality. Now we get by using Assertion 1

$$u_\nu=\sum_{\mu\in P}K_{\nu,\mu}(p+1,q+1)e^{\mu}=\sum_{\beta\in Q_+}K_{0,-\beta}(p+1,q+1)e^{-\beta}\times\sum_{\gamma\in P}K_{\nu,\gamma}e^{\gamma}=$$
$$\sum_{\beta,\gamma}K_{\nu,\gamma}K_{0,-\beta}(p+1,q+1)e^{\gamma-\beta}=\sum_{\mu\in P}\sum_{\beta\in Q_+}K_{\nu,\mu+\beta}K_{0,-\beta}(p+1,q+1)e^{\mu}$$

where we set $\mu=\gamma-\beta$ in the last equality. This gives

$$K_{\nu,\mu}(p+1,q+1)=\sum_{\beta\in Q_+}K_{\nu,\mu+\beta}K_{0,-\beta}(p+1,q+1)\in\mathbb{N}[p,q] \tag{14}$$

because the coefficients $K_{\nu,\mu+\beta}$ are nonnegative integers and we have already proved that the polynomials $K_{0,-\beta}(p+1,q+1)$ belongs to $\mathbb{N}[p,q]$. $\square$

Remark 4.2. Observe that in the decomposition (10) the sum is finite since the number of weights of the representation $V(\lambda)$ is and therefore $K_{\nu,\mu+\beta}$ is nonzero only for a finite number of weights β.

5. Stabilized version of the double deformation $K_{\nu,\mu}(p,q)$

5.1. **The stabilization phenomenon.** We resume the notation of Section 2. In particular, we recall the decomposition $\rho=\overline{\rho}+\rho^{\diamondsuit}$. Since $\rho=\sum_{i\in I}\omega_i$ and $\overline{\rho}=\sum_{i\in\overline{I}}\omega_i$, we must have

$$\rho^{\diamondsuit}=\sum_{i\in I\setminus\overline{I}}\omega_i.$$

The goal of this section is to prove that, once the two dominant weights ν and μ are fixed, the sequence of polynomials

$$K_{\nu+k\rho^{\diamondsuit},\mu+k\rho^{\diamondsuit}}(p,q),k\in\mathbb{N}$$

stabilizes and its limit has interesting positivity properties under a natural hypothesis inspired by the classical Lie algebras setting of types B_n,C_n,D_n and their parabolic restrictions to type A_{n-1} exploited in [12] and [13]. We will first need the following lemma.

Lemma 5.1. *Consider a dominant weight ν in P_+ and a weight $\mu\in P$ such that $\mu\leq\nu$. Then, there exists a nonnegative integer k_0 such that*

$$w(\nu+\rho+k\rho^{\diamondsuit})-(\mu+\rho+k\rho^{\diamondsuit})\notin Q_+$$

for any $k\geq k_0$ and any $w\in W\setminus\overline{W}$.

Proof. Consider $w\in W\setminus\overline{W}$. Since $\rho^{\diamondsuit}$ belongs to P_+ as a sum of fundamental weights, we must have $\rho^{\diamondsuit}-w(\rho^{\diamondsuit})\in Q_+$, that is $\rho^{\diamondsuit}-w(\rho^{\diamondsuit})$ is a nonnegative sum of positive roots in R_+. Moreover, $w(\rho^{\diamondsuit})\neq\rho^{\diamondsuit}$ because $w\in W\setminus\overline{W}$ and the stabilizer of $\rho^{\diamondsuit}$ under the action of W is be $\overline{W}=\langle s_i\mid i\in\overline{I}\rangle$. Therefore, we can set $\rho^{\diamondsuit}-w(\rho^{\diamondsuit})=\beta_w$ with $\beta_w\in Q_+\setminus\{0\}$. Now, we get for any nonnegative integer k

$$w(\nu+\rho+k\rho^{\diamondsuit})-(\mu+\rho+k\rho^{\diamondsuit})=w(\nu+\rho)-(\nu+\rho)+(\nu-\mu)-k\beta_w.$$

We have $w(\nu+\rho)-(\nu+\rho)<0$. Moreover, $\nu-\mu$ does not depends on k and $\beta_w>0$. There thus exists an integer k_w such that $w(\nu+\rho)-(\nu+\rho)+(\nu-\mu)-k\beta_w\notin Q_+$ for $k\geq k_w$ sufficiently large. We are done by considering $k_0=\max\{k_w\mid w\in W\setminus\overline{W}\}$ (recall W is finite). □

Proposition 5.2. *For any $\nu\in P_+$, any $\mu\in P$ and any $\overline{\lambda}+\gamma\in\overline{P}_+$ such that $\overline{\lambda}+\gamma\geq\mu$, there exists a nonnegative integer k_0 such that for any $k\geq k_0$*

$$b_{\nu+k\rho^\diamondsuit,\overline{\lambda}+\gamma+k\rho^\diamondsuit}(p)=b_{\lambda+k_0\rho^\diamondsuit,\overline{\lambda}+\gamma+k_0\rho^\diamondsuit}(p)=\sum_{\overline{w}\in\overline{W}}\varepsilon(\overline{w})\hat{\mathrm{P}}_p(\overline{w}(\nu+\overline{\rho})-(\overline{\lambda}+\gamma+\overline{\rho})).$$

In particular, $b_{\nu+k\rho^\diamondsuit,\overline{\lambda}+\gamma+k\rho^\diamondsuit}(p)$ does not depend on k for $k\geq k_0$ and we can set in this case $b^{\mathrm{stab}}_{\nu,\overline{\lambda}+\gamma}(p):=b_{\nu+k\rho^\diamondsuit,\overline{\lambda}+\gamma+k\rho^\diamondsuit}(p)$.

Proof. By Lemma 5.1, there exists a nonnegative integer k_0 such that for any $k\geq k_0$ and any $w\in W\setminus\overline{W}$, we have $w(\nu+\rho+k\rho^\diamondsuit)-(\mu+\rho+k\rho^\diamondsuit)\notin Q_+$. Since we then have $\overline{\lambda}+\gamma\geq\mu$ we also obtain that

$$w(\nu+\rho+k\rho^\diamondsuit)-(\rho+\overline{\lambda}+\gamma+k\rho^\diamondsuit)=w(\nu+\rho+k\rho^\diamondsuit)-(\mu+\rho+k\rho^\diamondsuit)-(\overline{\lambda}+\gamma-\mu)\notin Q_+$$

This gives

$$b_{\nu+k\rho^\diamondsuit,\overline{\lambda}+\gamma+k\rho^\diamondsuit}(p)=\sum_{w\in W}\varepsilon(w)\hat{\mathrm{P}}_p(w(\nu+\overline{\rho}+k\rho^\diamondsuit)-(\overline{\lambda}+\gamma+\overline{\rho}+k\rho^\diamondsuit))=$$
$$\sum_{\overline{w}\in\overline{W}}\varepsilon(\overline{w})\hat{\mathrm{P}}_p(\overline{w}(\nu+\overline{\rho}+k\rho^\diamondsuit)-(\overline{\lambda}+\gamma+\overline{\rho}+k\rho^\diamondsuit))=\sum_{\overline{w}\in\overline{W}}\varepsilon(\overline{w})\hat{\mathrm{P}}_p(\overline{w}(\nu+\overline{\rho})-(\overline{\lambda}+\gamma+\overline{\rho}))$$

because $\hat{\mathrm{P}}_p(\beta)=0$ for $\beta\notin Q_+$ and $\overline{w}(\rho^\diamondsuit)=\rho^\diamondsuit$ for any $\overline{w}$ in $\overline{W}$. □

Remark 5.3. In [16], we establish a nonnegative expansion of

$$\prod_{\alpha\in R_+\setminus\overline{R}_+}\frac{1}{1-pe^\alpha}$$

in $\overline{\mathfrak{g}}$-Weyl characters from which it becomes possible to deduce that the stabilized polynomials $b^{\mathrm{stab}}_{\nu,\mu}(p)$ belong to $\mathbb{N}[p]$ for any $\nu\in P_+$ and any $\mu=\mu^\diamondsuit\in\overline{P}_+$ (see Prop 3.9 in [16]).

Theorem 5.4. *For any dominant weight $\nu\in P_+$ and any weight μ in P there exists a nonnegative integer k_0 such that, for any $k\geq k_0$ we have the decomposition*

$$K_{\nu+k\rho^\diamondsuit,\mu+k\rho^\diamondsuit}(p,q)=\sum_{\overline{\lambda}\in\overline{P}_+}b^{\mathrm{stab}}_{\nu,\overline{\lambda}+\mu^\diamondsuit}(p)\overline{K}_{\overline{\lambda},\overline{\mu}}(q).$$

In particular, the stabilized double deformation $K_{\nu+k\rho^\diamondsuit,\mu+k\rho^\diamondsuit}(p,q)=K^{\mathrm{stab}}_{\nu,\mu}(p,q)$ does not depend on $k\geq k_0$.

Proof. By using Lemma 5.1 and the definition (4) of $K_{\nu+k\rho^\diamondsuit,\mu+k\rho^\diamondsuit}(p,q)$, we get

$$K_{\nu+k\rho^\diamondsuit,\mu+k\rho^\diamondsuit}(p,q)=\sum_{\overline{w}\in\overline{W}}\varepsilon(\overline{w})\mathrm{P}_{p,q}(\overline{w}(\nu+\rho)-\mu-\rho).$$

Indeed, by using the integer k_0 of Lemma 5.1, for $k\geq k_0$, we have for any $w\in W\setminus\overline{W}$

$$w(\nu+\rho+k\rho^\diamondsuit)-\mu-\rho-k\rho^\diamondsuit\notin Q_+$$

and then $\mathrm{P}_{p,q}(\overline{w}(\nu+\rho+k\rho^\diamondsuit)-\mu-\rho-k\rho^\diamondsuit)=0$. In the remaining cases

$$\overline{w}(\nu+\rho+k\rho^\diamondsuit)-\mu-\rho-k\rho^\diamondsuit=\overline{w}(\nu+\rho)-\mu-\rho$$

since $\overline{w}(\rho^{\diamondsuit}) = \rho^{\diamondsuit}$. Now, we can argue exactly as in the proof of Theorem 3.1 (but by replacing the set U by $\{id\}$). The equality (8) is then replaced by

$$K_{\nu,\mu}(p,q) = \sum_{\overline{\lambda}\in\overline{P}_+} \left(\sum_{\overline{w}\in\overline{W}} \varepsilon(\overline{w}) \hat{\mathrm{P}}_p \left(\overline{w}(\nu+\rho) - \mu^{\diamondsuit} - \overline{\lambda} - \rho \right) \right) \overline{K}_{\overline{\lambda},\overline{\mu}}(q) = \sum_{\overline{\lambda}\in\overline{P}_+} b^{\mathrm{stab}}_{\nu,\overline{\lambda}+\mu^{\diamondsuit}}(p) \overline{K}_{\overline{\lambda},\overline{\mu}}(q)$$

where the last equality follows from Proposition 5.2 with $\gamma = \mu^{\diamondsuit}$ and $\overline{\lambda} + \mu^{\diamondsuit} \geq \mu$ since $\overline{\lambda} \geq \overline{\mu}$ when $\overline{K}_{\overline{\lambda},\overline{\mu}}(q) \neq 0$. □

Remark 5.5.

(1) When $p = q$, the previous result contains, as a particular case, the stabilization of the Lusztig q-analogues of classical types B_n, C_n and D_n by translation of the two dominant weights ν, μ by multiples of the fundamental weight ω_n. This case where $\rho^{\diamondsuit} = \omega_n$ corresponds to the choice of the parabolic subsystem of type A_{n-1} obtained by removing the node n in the Dynkin diagrams and was considered in particular in [15] and [4].

(2) In Lemma 5.1, Proposition 5.2 and Theorem 5.4, the crucial argument is the invariance of $\rho^{\diamondsuit}$ under the action of $\overline{W}$. Instead of choosing first the parabolic root system and then the direction $\rho^{\diamondsuit}$ in the Weyl chamber of $\mathfrak{g}$, it is also interesting to first consider a weight δ in P_+ and next the set $\overline{I} := \{i \in I \mid \langle \delta, \alpha_i \rangle = 0\}$. Then $\overline{W}$ stabilizes δ. By using the same arguments, we can then establish the theorem below, companion of Theorem 5.4 where the decomposition of δ on the basis of fundamental weights can make appear multiplicities greater than one.

Theorem 5.6. *Assume that $\overline{I}$ is chosen as previously from the dominant weight δ. Then for any weight μ in P there exists a nonnegative integer k_0 such that, for any $k \geq k_0$ we have the decomposition*

$$K_{\nu+k\delta,\mu+k\delta}(p,q) = \sum_{\overline{\lambda}\in\overline{P}_+} b^{\delta\text{-stab}}_{\nu,\overline{\lambda}+\mu^{\diamondsuit}}(p) \overline{K}_{\overline{\lambda},\overline{\mu}}(q)$$

where $b^{\delta\text{-stab}}_{\nu,\overline{\lambda}+\gamma}(p) := b_{\nu+k\delta,\overline{\lambda}+\gamma+k\delta}(p)$. In particular, the stabilized double deformation $K^{\delta\text{-stab}}_{\nu,\mu}(p,q) = K_{\nu+k\delta,\mu+k\delta}(p,q)$ does not depend on $k \geq k_0$.

5.2. **Positivity of the stabilized forms.** In this paragraph, we shall need the following additional hypothesis.

Hypothesis 5.7. We assume in this section that $\langle \rho^{\diamondsuit}, \alpha \rangle = c$ is a fixed positive integer for any $\alpha \in R_+ \setminus \overline{R}_+$ which does not depend on the choice of α in $R_+ \setminus \overline{R}_+$.

Remark 5.8.

(1) Observe that our condition is in particular satisfied for the classical root system of type C_n or D_n when $\overline{I} = \{1, \ldots, n-1\}$ is the set of labels of the parabolic subsystem of type A_{n-1}. Then for any $\alpha \in R_+ \setminus \overline{R}_+$, we have $\langle \rho^{\diamondsuit}, \alpha \rangle = \langle \omega_n, \alpha \rangle = 2$ in type C_n and $\langle \rho^{\diamondsuit}, \alpha \rangle = \langle \omega_n, \alpha \rangle = 1$ in type D_n.

(2) It is also satisfied for the parabolic subsystems of type B_{n-1} or D_{n-1} inside the root systems of type B_n or D_n, respectively. We then indeed get $\langle \rho^{\diamondsuit}, \alpha \rangle = \langle \omega_1, \alpha \rangle = 1$ for any $\alpha \in R_+ \setminus \overline{R}_+$.

(3) Another case where the hypothesis is satisfied if for $\mathfrak{g}$ of type A_{n-1} and $\overline{\mathfrak{g}}$ the Levi subalgebra obtained by removing the k-th node in the Dynkin diagram of $\mathfrak{g}$. In this case, we obtain $\langle \rho^{\diamondsuit}, \alpha \rangle = \langle \omega_k, \alpha \rangle = 1$ for any $\alpha \in R_+ \setminus \overline{R}_+$.

(4) As in [15], it could be also interesting to extend the stabilization phenomenon to deformations of the Lusztig analogues defined by using weight functions on p and q depending on the length of the root considered. When $p = q$, this is in particular unavoidable for equating the one-dimensional sums for affine root systems with q-analogues of weight multiplicities. For simplicity, we do not pursue in this direction here.

Consider γ in $\hat{Q}_+$ and a decomposition $\gamma = \sum_{\alpha\in R_+\setminus\overline{R}_+} a_\alpha \alpha$ where the a_α's are nonnegative integers. Then

$$L(\gamma) := \frac{1}{c}\langle \rho^\diamondsuit, \gamma\rangle = \frac{1}{c}\sum_{\alpha\in R_+\setminus\overline{R}_+} a_\alpha \langle \rho^\diamondsuit, \alpha\rangle = \sum_{\alpha\in R_+\setminus\overline{R}_+} a_\alpha$$

by Hypothesis 5.7. This shows that the number of positive roots in $R_+ \setminus \overline{R}_+$ appearing in a relevant decomposition of γ is always equal to $L(\gamma)$. Therefore, we have $\hat{\mathrm{P}}_p(\gamma) = p^{L(\gamma)}\hat{\mathrm{P}}_1(\gamma)$ for any γ in $\hat{Q}_+$. Moreover, we have by Proposition 5.2

$$b^{\mathrm{stab}}_{\nu,\overline{\lambda}+\mu^\diamondsuit}(q) = \sum_{\overline{w}\in\overline{W}} \varepsilon(\overline{w})\hat{\mathrm{P}}_p(\overline{w}(\nu+\overline{\rho}) - (\overline{\lambda} + +\mu^\diamondsuit + \overline{\rho})).$$

For any $\overline{w} \in \overline{W}$, and with $\gamma = \overline{w}(\nu+\overline{\rho}) - (\overline{\lambda} + \mu^\diamondsuit + \overline{\rho})$

$$\begin{aligned} L(\gamma) = \frac{1}{c}\langle \rho^\diamondsuit, \overline{w}(\nu+\overline{\rho}) - (\overline{\lambda}+\mu^\diamondsuit+\overline{\rho})\rangle = \frac{1}{c}\langle \rho^\diamondsuit, \overline{w}(\nu+\overline{\rho})\rangle - \frac{1}{c}\langle \rho^\diamondsuit, \overline{\lambda}+\mu^\diamondsuit+\overline{\rho}\rangle = \\ \frac{1}{c}\langle \overline{w}^{-1}(\rho^\diamondsuit), \nu+\overline{\rho}\rangle - \frac{1}{c}\langle \rho^\diamondsuit, \overline{\lambda}+\mu^\diamondsuit+\overline{\rho}\rangle = \frac{1}{c}\langle \rho^\diamondsuit, \nu+\overline{\rho}\rangle - \frac{1}{c}\langle \rho^\diamondsuit, \overline{\lambda}+\mu^\diamondsuit+\overline{\rho}\rangle = \\ \frac{1}{c}\langle \rho^\diamondsuit, \nu - \overline{\lambda} - \mu^\diamondsuit\rangle = L(\nu - \overline{\lambda} - \mu^\diamondsuit) \end{aligned}$$

because $\rho^\diamondsuit$ is stabilized by $\overline{W}$. Therefore

$$b^{\mathrm{stab}}_{\nu,\overline{\lambda}}(p) = p^{L(\nu-\overline{\lambda}-\mu^\diamondsuit)} b^{\mathrm{stab}}_{\nu,\overline{\lambda}}(1) \text{ for any } \nu \in P_+ \text{ and any } \overline{\lambda} \in \overline{P}_+.$$

This proves that $b^{\mathrm{stab}}_{\nu,\overline{\lambda}}(q)$ has nonnegative integer coefficients. We then get the following Corollary of Theorem 5.4.

Corollary 5.9. *Under Hypothesis 5.7, for any dominant weights $\nu \in P_+$ and $\mu \in \overline{P}_+$, the polynomial $K^{\mathrm{stab}}_{\nu,\mu}(p,q)$ belongs to $\mathbb{N}[p,q]$. Moreover, we have*

$$K_{\nu+k\rho^\diamondsuit,\mu+k\rho^\diamondsuit}(p,q) = \sum_{\overline{\lambda}\in\overline{P}_+} p^{L(\nu-\overline{\lambda}-\mu^\diamondsuit)} b^{\mathrm{stab}}_{\nu,\overline{\lambda}+\mu^\diamondsuit}(1)\overline{K}_{\overline{\lambda},\overline{\mu}}(q).$$

Example 5.10. Resume the example of type C_2 in Remark 3.4. For $\nu = (4,2)$ and $\mu = (0,0)$, we had

$$K_{(4,2),(0,0)}(p,q) = p^3q^4 + p^3q^3 + 2p^3q^2 + p^3q + p^3 - q^4.$$

Here $\rho^\diamondsuit = \omega_2 = (1,1)$. We get

$$K_{(5,3),(1,1)}(p,q) = p^3q^4 + p^3q^3 + 2p^3q^2 + p^3q + p^3$$

with nonnegative integer coefficients and

$$K_{\nu+k\omega_2,\mu+k\omega_2}(p,q) = K_{(5,3),(1,1)}(p,q)$$

for any $k > 0$.

Remark 5.11. Here again, one can follow the same line as in Remark 5.5 and consider a dominant weight δ in P_+ with $\overline{I} := \{i \in I \mid \langle \delta, \alpha_i \rangle = 0\}$. Then replace Hypothesis 5.7 by the condition

$$\langle \delta, \alpha \rangle = c \tag{15}$$

is a positive constant for any $\alpha \in R_+ \setminus \overline{R}_+$ and set $L_\delta(\gamma) := \frac{1}{c}\langle \delta, \gamma \rangle$ for any $\gamma = \sum_{\alpha \in R_+ \setminus \overline{R}_+} a_\alpha \alpha$. This gives the following companion corollary of Theorem 5.6.

Corollary 5.12. *Assume δ is fixed in P_+ and then the convention and hypotheses of the previous remark. Then, for any dominant weights $\nu \in P_+$ and $\mu \in \overline{P}_+$, the polynomial $K_{\nu,\mu}^{\delta\text{-stab}}(p,q)$ belongs to $\mathbb{N}[p,q]$. Moreover, we have*

$$K_{\nu,\mu}^{\delta\text{-stab}}(p,q) = K_{\nu+k\delta,\mu+k\delta}(p,q) = \sum_{\overline{\lambda} \in \overline{P}_+} p^{L_\delta(\nu - \overline{\lambda} - \mu^\diamondsuit)} b_{\nu,\overline{\lambda}+\mu^\diamondsuit}^{\delta\text{-stab}}(1) \overline{K}_{\overline{\lambda},\overline{\mu}}(q).$$

6. Combinatorial description

6.1. **Combinatorial description of the double deformation $K_{\nu,\mu}(p,q)$.** A difficult question in the theory of Lusztig q-analogues of weight multiplicities consists in their combinatorial description. In type A_{n-1} this was achieved by Lascoux and Schützenberger in [11]. For the classical types, the description is only partially known. We refer the interested reader to [15] (where it is done by equating them with affine 1-d sums in the stable case) and [4] (where such an equality is proved for the general cases in type C_n and B_n with half integer weights), [14], and [17] where a description is given for $\mu = 0$ respectively in symplectic and orthogonal types by using generalized exponents. In view to Theorem 3.1, the combinatorial description of the polynomials $K_{\nu,\mu}(p,q)$ reduces to the combinatorial description of the polynomials $b_{\nu,\overline{\lambda}+\mu^\diamondsuit}(p)$ and $\overline{K}_{\overline{\lambda},\overline{\mu}}(q)$.

For $p = 1$, the branching coefficient $b_{\nu,\overline{\lambda}+\mu^\diamondsuit}(1)$ can be easily described thanks to crystal graph theory or Littelmann path theory since both are compatible with parabolic restrictions. Recall that to each dominant weight ν, is associated a crystal $B(\nu)$ which can be regarded as the combinatorial skeleton of the representation $V(\nu)$. This is an oriented graph where the arrows are described in terms of the crystal operators $\tilde{f}_i$ and $\tilde{e}_i$ with $i \in I$. Each vertex b in $B(\nu)$ has a weight $\mathrm{wt}(b) \in P$ so that

$$s_\nu = \sum_{b \in B(\nu)} e^{\mathrm{wt}(b)}. \tag{16}$$

In particular the cardinality of $B(\nu)$ is equal to the dimension of $V(\nu)$. Also $B(\nu)$ has a unique source vertex b_λ which is the unique vertex killed by all the crystal operators $\tilde{e}_i, i \in I$. The multiplicity $b_{\nu,\overline{\lambda}+\mu^\diamondsuit}(1)$ is then equal to the cardinality of the set

$$H_{\overline{\lambda}+\mu^\diamondsuit} = \{b \in B(\nu) \mid \tilde{e}_i(b) = 0, i \in \overline{I} \text{ and } \mathrm{wt}(b) = \overline{\lambda} + \mu^\diamondsuit\}.$$

This thus reduces the combinatorial description of the polynomials $K_{\nu,\mu}(1,q)$ to that of the polynomials $\overline{K}_{\overline{\lambda},\overline{\mu}}(q)$, i.e. to that of the ordinary Lusztig q-analogues. More precisely, denote by $\overline{B}(b)$ the parabolic crystal with highest weight vertex $b \in H_{\overline{\lambda}+\mu^\diamondsuit}$ (that is, the crystal $\overline{B}(b)$ is obtained by applying the crystal operators $\tilde{f}_i, i \in \overline{I}$ to b). Then $\mathrm{wt}(b)$ belongs to $\overline{P}_+$. Assume one knows a map

$$\overline{\mathrm{ch}} : \overline{B}(b) \to \mathbb{N} \tag{17}$$

such that

$$\sum_{b'\in\overline{B}(b)_\mu} q^{\overline{\mathrm{ch}}(b')} = \overline{K}_{\overline{\lambda}+\mu^\diamond,\mu}(q) = \overline{K}_{\overline{\lambda},\overline{\mu}}(q)$$

where $\mathrm{wt}(b) = \overline{\lambda} + \mu^\diamond$ belongs to $\overline{P}_+$ and $\overline{B}(b)_\mu = \{b' \in \overline{B}(b) \mid \mathrm{wt}(b') = \mu\}$. Then, we have by Theorem 3.1

$$K_{\nu,\mu}(1,q) = \sum_{b\in H_{\overline{\lambda}+\mu^\diamond}} \sum_{b'\in\overline{B}(b)_\mu} q^{\overline{\mathrm{ch}}(b')}.$$

In particular, when $\mathfrak{g}$ is of classical type B_n, C_n or D_n and $\overline{\mathfrak{g}}$ is isomorphic to $\mathfrak{gl}_n$ (i.e. $\overline{I} = I\setminus\{n\}$), there are various combinatorial models for the crystal $B(\nu)$ and also for computing the branching coefficients $b_{\nu,\overline{\lambda}+\mu^\diamond}(1)$ without constructing the whole crystal $B(\nu)$ (see [10] and the reference therein). Moreover the parabolic Lusztig q-analogues $\overline{K}_{\overline{\lambda},\overline{\mu}}(q)$ then coincide with the Kostka polynomials of type A_{n-1} whose combinatorial description was given by Lascoux and Schützenberger in [11] thanks to the so-called charge statistics on semistandard tableaux (see also [23] for a recent geometric description of the charge). This charge statistics can also be defined purely in terms of crystals and then be used as a map $\overline{\mathrm{ch}}$ (see (17)). For the exceptional types, there is always a unique node in I to remove in order to get a maximal parabolic root system of type A. Therefore, this strategy also works even if the combinatorial description of the crystals is then more complicated.

6.2. **Combinatorial description of the double deformation $K_{\nu,\mu}(p+1,q+1)$.** We will now explain how it is possible to get a combinatorial description of the polynomials $K_{\nu,\mu}(p+1,q+1)$ when $\nu \in P_+$ and $\mu \in P$ in terms of a polynomial statistics on the vertices of the crystal graph $B(\nu)$ thanks to Theorem 4.1. As far as we are aware this combinatorial description is new even in the case where $p = q$. To do this, let us consider two copies R_+^b and R_+^r of the set of positive roots R_+ that can be thought as positive roots colored in blue and red. Let $\mathcal{R}_+ = R_+^b \sqcup R_+^r$. We will use bold symbols to denote a colored positive root $\boldsymbol{\alpha}$ in $\mathcal{R}_+$ and write $|\boldsymbol{\alpha}|$ for its corresponding uncolored root in R_+ (obtained by forgetting its color). For any multiset S of $\mathcal{R}_+$ (this means that a colored positive root may appear more than once in S), denote by $d(S)$ and $e(S)$ respectively the numbers of blue roots in $R_+^b \setminus \overline{R}_+^b$ and $\overline{R}_+^b$ it contains. Write for short $S \Subset \mathcal{R}_+$ when S is a multiset of $\mathcal{R}_+$ and for such a multiset, put $\Sigma(S) = \sum_{\alpha\in S} |\boldsymbol{\alpha}|$ the sum of the roots it contains once the colors are forgotten.

Lemma 6.1. *We have*

$$\frac{1}{\prod_{\alpha\in R_+\setminus\overline{R}_+}(1-(p+1)e^{-\alpha})\prod_{\alpha\in\overline{R}_+}(1-(q+1)e^{-\alpha})} = \sum_{\beta\in Q_+} N_{p,q}(\beta)e^{-\beta} \tag{18}$$

where

$$N_{p,q}(\beta) = \sum_{S\Subset\mathcal{R}_+\mid\Sigma(S)=\beta} p^{d(S)}q^{e(S)}. \tag{19}$$

Proof. We have for $\alpha \in R_+ \setminus \overline{R}_+$

$$\frac{1}{1-(p+1)e^{-\alpha}} = \sum_{k\geq 0}(p+1)^k e^{-k\alpha} = \sum_{k\geq 0}\sum_{a=0}^{k}\binom{k}{a}p^a e^{-k\alpha} = \sum_{S_\alpha\Subset\{\boldsymbol{\alpha}^b,\boldsymbol{\alpha}^r\}} p^{d(S_\alpha)}e^{-\Sigma(S_\alpha)}$$

where the last sum runs over the multisets S_α containing only the colored roots $\boldsymbol{\alpha}^b$ or $\boldsymbol{\alpha}^r$. Similarly for $\alpha \in \overline{R}_+$, one gets

$$\frac{1}{1-(q+1)e^{-\alpha}} = \sum_{S_\alpha \Subset \{\boldsymbol{\alpha}^b, \boldsymbol{\alpha}^r\}} q^{e(S_\alpha)} e^{-\Sigma(S_\alpha)}.$$

The lemma follows by using that the left hand side of (18) is product rational geometric series of the two previous types. □

Now recall we proved in Theorem 4.1 the identity

$$\Delta = \frac{\prod_{\alpha \in R_+} (1-e^{-\alpha})}{\prod_{\alpha \in R_+ \setminus \overline{R}_+} (1-(p+1)e^{-\alpha}) \prod_{\alpha \in \overline{R}_+} (1-(q+1)e^{-\alpha})} = \sum_{\beta \in Q_+} K_{0,-\beta}(p+1, q+1) e^{-\beta}.$$

In order to get an analogue of Lemma 6.1 for Δ, we need to restrict the possible multisets of $\mathcal{R}_+$ that we consider. Given a colored positive root $\boldsymbol{\alpha}$ and $S \Subset \mathcal{R}$, denote by $m_{\boldsymbol{\alpha}}(S)$ its multiplicity in S. Let us say that $S \Subset \mathcal{R}_+$ is admissible when for any red root $\boldsymbol{\alpha} \in R_+^b$ such that $m_{\boldsymbol{\alpha}}(S) > 0$, its associated blue root $\boldsymbol{\alpha}'$ such that $|\boldsymbol{\alpha}| = |\boldsymbol{\alpha}'|$ also satisfies $m_{\boldsymbol{\alpha}'}(S) > 0$ (but we do not assume here that $m_{\boldsymbol{\alpha}}(S) = m_{\boldsymbol{\alpha}'}(S)$). Roughly speaking, this means that S is admissible when it cannot contain a red positive root $\boldsymbol{\alpha}$ such that $\alpha = |\boldsymbol{\alpha}|$ without also containing the blue version of α. Observe that the empty set is admissible. Write for short $S \Subset_a \mathcal{R}_+$ when S is an admissible multiset of $\mathcal{R}_+$. Finally for any β in Q set

$$R_{p,q}(\beta) = \sum_{S \Subset_a \mathcal{R}_+ \mid \Sigma(S) = \beta} p^{d(S)} q^{e(S)} \tag{20}$$

when $\beta \in Q_+$ and $R_{p,q}(\beta) = 0$ otherwise. We get the following equality.

Lemma 6.2. *For ant β in Q_+ we have $K_{0,-\beta}(p+1, q+1) = R_{p,q}(\beta)$.*

Proof. The proof follows the same line as Proof of Lemma 19. As in (12), we get for $\alpha \in R_+ \setminus \overline{R}_+$

$$\frac{1-e^{-\alpha}}{1-(p+1)e^{-\alpha}} = 1 + pe^{-\alpha} \sum_{k \geq 0} (p+1)^k e^{-k\alpha} = 1 + pe^{-\alpha} \sum_{k \geq 0} \sum_{a=0}^{k} \binom{k}{a} p^a e^{-k\alpha} = \sum_{S_\alpha \Subset_a \{\boldsymbol{\alpha}^b, \boldsymbol{\alpha}^r\}} p^{d(S_\alpha)} e^{-\Sigma(S_\alpha)}$$

where the last sum runs over the restricted multisets S_α containing only the colored roots $\boldsymbol{\alpha}^b$ or $\boldsymbol{\alpha}^r$. Similarly for $\alpha \in \overline{R}_+$, one gets

$$\frac{1-e^{-\alpha}}{1-(q+1)e^{-\alpha}} = \sum_{S_\alpha \Subset_r \{\boldsymbol{\alpha}^b, \boldsymbol{\alpha}^r\}} q^{e(S_\alpha)} e^{-\Sigma(S_\alpha)}.$$

The lemma follows by using that Δ is a product of rational series of the two previous types. □

We now define the polynomial μ-charge statistics $\chi_{p,q}^\mu$ on the vertices of $B(\lambda)$ as follows

$$\chi_{p,q}^\mu : \left\{ \begin{array}{c} B(\lambda) \to \mathbb{Z}_{\geq 0}[p,q] \\ b \longmapsto R_{p,q}(\mathrm{wt}(b) - \mu) \end{array} \right.$$

where the polynomials $R_{p,q}$ are defined in (20) and the weight function wt is the same as in (16). In particular, we have $\chi_{p,q}^\mu(b) = 0$ when $\mathrm{wt}(b)$ is not greater or equal to μ in the dominant order. The following theorem describes the polynomial $K_{\nu,\mu}(p+1,q+1)$ as the generating series for the polynomial μ-charge statistics $\chi_{p,q}$ over the vertices of the crystal $B(\nu)$.

Theorem 6.3. *For any dominant weight ν and any weight μ, we have*

$$K_{\nu,\mu}(p+1,q+1) = \sum_{b\in B(\nu)_{\geq\mu}} \chi^{\mu}_{p,q}(b)$$

where $B(\nu)_{\geq\mu} = \{b \in B(\nu) \mid \mathrm{wt}(b) \geq \mu\}$.

Proof. By Theorem 4.1 and Lemma 6.2, one obtains the decomposition

$$K_{\nu,\mu}(p+1,q+1) = \sum_{\beta\in Q_+} K_{\nu,\mu+\beta}K_{0,-\beta}(p+1,q+1) = \sum_{\beta\in Q_+} K_{\nu,\mu+\beta}R_{p,q}(\beta).$$

Now, we have

$$K_{\nu,\mu+\beta} = \mathrm{card}\,(b \in B(\nu) \mid \mathrm{wt}(b) = \mu+\beta)$$

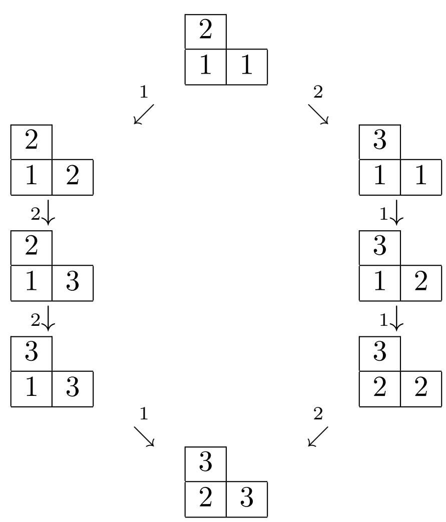
and therefore

$$K_{\nu,\mu}(p+1,q+1) = \sum_{\beta\in Q_+}\ \sum_{b\in B(\nu)\mid \mathrm{wt}(b)=\mu+\beta} R_{p,q}(\beta) = \sum_{b\in B(\nu)} R_{p,q}(\mathrm{wt}(b)-\mu) =$$

$$\sum_{b\in B(\nu)} \chi^{\mu}_{p,q}(b) = \sum_{b\in B(\nu)_{\geq\mu}} \chi^{\mu}_{p,q}(b)$$

as desired since $\chi^{\mu}_{p,q}(b) = 0$ when $\mathrm{wt}(b)$ is not greater or equal to μ. □

Example 6.4. We give below the tableau realization of the crystal graph $B(\lambda)$ of the adjoint representation in type A_2 with highest weight $\lambda = \omega_1 + \omega_2$. We have $R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2)$. For $\mu = 0$ we indicate the values of the polynomial statistics $\chi^{\mu}_{p,q}(b)$ on the vertices b such that $\mathrm{w}(b) \geq 0$:



b	$\begin{smallmatrix}2 & \\ 1 & 3\end{smallmatrix}$	$\begin{smallmatrix}3 & \\ 1 & 2\end{smallmatrix}$	$\begin{smallmatrix}2 & \\ 1 & 2\end{smallmatrix}$	$\begin{smallmatrix}3 & \\ 1 & 1\end{smallmatrix}$	$\begin{smallmatrix}2 & \\ 1 & 1\end{smallmatrix}$
$\chi^{\mu}_{p,p}(b)$	1	1	q	q	q^2+q

For example, when $b = \begin{smallmatrix}2 & \\ 1 & 1\end{smallmatrix}$ is the highest weight vertex, the admissible multisets of colored roots summing up $\lambda = \alpha_1 + \alpha_2$ are

$$\{(\alpha_1 + \alpha_2)_b\} \text{ and } \{\{(\alpha_1)_b, \{\alpha_2)_b\}$$

which gives $\chi^{\mu}_{p,p}(b) = q^2 + q$. Finally we have

$$K_{\lambda,\mu}(q+1, q+1) = 2 + 2q + (q^2 + q) = q^2 + 3q + 2 = (q+1)^2 + (q+1)$$

as expected.

Remark 6.5. One could imagine obtaining a combinatorial description of the Lusztig q-analogue $K_{\lambda,\mu}(q,q)$ from that of $K_{\lambda,\mu}(q+1, q+1)$ by changing q in $q-1$ and next trying to understand the possible cancellations. This interesting question does not seem easy in particular because $K_{\lambda,\mu}(q,q)$ only have nonnegative coefficients when both λ and μ are dominant.

6.3. **Some analogues of the Hall-Littlewood polynomials.** Recall that the character ring $\mathbb{Z}[p,q]^W[P]$ of $\mathfrak{g}$ can be endowed with a scalar product $\langle \cdot, \cdot \rangle_{(p,q)}$ such that $\langle s_\nu, s_\mu \rangle_{(p,q)} = \delta_{\nu,\mu}$ for any dominant weights ν and μ in P_+. For any μ in P_+, one can then define the polynomial

$$Q'_\mu(p,q) = \frac{1}{a_\rho} \sum_{w \in W} \varepsilon(w) w \left(\frac{e^{\mu+\rho}}{\prod_{\alpha \in R_+ \setminus \overline{R}_+} (1 - p e^{\alpha}) \prod_{\alpha \in \overline{R}_+} (1 - q e^{\alpha})} \right).$$

Proposition 6.6. *We have*

$$Q'_\mu(p,q) = \sum_{\nu \in P_+, \mu \leq \nu} K_{\nu,\mu}(p,q) s_\nu.$$

In particular $\{Q'_\mu(p,q) \mid \mu \in P_+\}$ *is a* $\mathbb{Z}$*-basis of* $\mathbb{Z}[p,q]^W[P]$.

Proof. By using the expansion (3) of the (p,q)-Kostant partition, we get

$$Q'_\mu(p,q) = \sum_{\beta \in Q_+} \frac{1}{a_\rho} \sum_{w \in W} \varepsilon(w) \mathrm{P}_{p,q}(\beta) e^{w(\mu+\rho+\beta)} = \sum_{\beta \in Q_+} \mathrm{P}_{p,q}(\beta) \frac{a_{\mu+\beta+\rho}}{a_\rho} = \sum_{\beta \in Q_+} \mathrm{P}_{p,q}(\beta) s_{\mu+\beta}.$$

Now, for any $\beta \in Q_+$ we have $s_{\mu+\beta} = 0$ or their exists an element w in W and a dominant weight ν such that $\nu + \rho = w^{-1}(\mu + \beta + \rho)$ and $s_{\mu+\beta} = \varepsilon(w) s_\nu$. Then, we have $\beta = w(\nu+\rho) - \rho - \mu$. By gathering the contributions of the same character s_ν, one obtains

$$Q'_\mu(p,q) = \sum_{w \in W} \varepsilon(w) \mathrm{P}_{p,q}(w(\nu+\rho) - \rho - \mu) s_\nu = \sum_{\nu \in P_+, \mu \leq \nu} K_{\nu,\mu}(p,q) s_\nu$$

as expected because $K_{\nu,\mu}(p,q)$ is nonzero only when $\mu \leq \nu$. □

We can then define the basis $\{P_\nu(p,q) \mid \nu \in P_+\}$ as the dual basis of $\{Q'_\mu(p,q) \mid \mu \in P_+\}$ for the scalar product $\langle \cdot, \cdot \rangle_{(p,q)}$. We then have

$$s_\nu = \sum_{\mu \in P_+} K_{\nu,\mu}(p,q) P_\mu(p,q).$$

When $p = q$, the polynomials $P_\mu(q)$ are the Hall-Littlewood polynomials. They are deeply connected with the theory of affine Hecke algebras (see [20] for a detailed review). In particular, by using the Satake isomorphism, one can prove that the polynomials $K_{\nu,\mu}(q,q) = K_{\nu,\mu}(q)$ are affine Kazhdan-Lusztig polynomials. For the double deformations introduced here, the connection with the theory of affine Hecke algebras cannot be obtained similarly. There indeed exist unequal parameters affine Hecke algebras constructed from weight functions on the set of positive roots. Nevertheless, these weight functions should be constant on the roots which belong to the same orbit under the action of W. This is not always the case in our setting since $\overline{R}_+$ is not stable under the action of W.

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Cédric Lecouvey: Institut Denis Poisson Tours.
Université de Tours Parc de Grandmont, 37200 Tours, France.
cedric.lecouvey@univ-tours.fr

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