

Symmetric Spaces with Rectangular Unit Lattices, Revisited

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Abstract. In this paper we study compact Riemannian symmetric spaces whose maximal tori are as simple as possible: metric products of circles. In particular, using representation theory, we construct explicitly an isometric embedding of such spaces X into Euclidean space with the property that geodesic symmetries of X extend to reflections along normal spaces. The converse is also true: extrinsically symmetric submanifolds have rectangular maximal tori. Further, such embedding of X is unique up to congruence, and all isometries of X extend to isometries of the ambient Euclidean space.

1. Introduction

This work grew out from an attempt to understand a fundamental theorem of LOOS, namely [L3, Satz 6], announced in [L2] and saying that a Riemannian symmetric space with cubic unit lattice is affinely equivalent to a symmetric R -space. Recall that a *symmetric R -space* arises as an orbit $L(\xi)$ of a point ξ in the boundary at infinity of a Riemannian symmetric space L/G of non-compact type (L semi-simple, non-compact and with trivial center and G maximal compact in L) such that (G, G_ξ) is a symmetric pair, where G_ξ is the stabilizer of ξ in G . Since $G(\xi) = L(\xi)$, the symmetric R -space can be identified with G/G_ξ and is thus a compact affine symmetric space. The *unit lattice* of a compact Riemannian symmetric space X is the lattice of a maximal torus T of X that is the set of all tangent vectors of T at a point $p \in T$ which are mapped to p by the Riemannian exponential map at p . It is called *rectangular (cubic)* if it has an orthogonal basis (orthonormal basis after scaling).

LOOS' proof of [L3, Satz 6] consists of translating the problem into the language of Jordan triple systems where he solves it using extensive calculations. We were therefore looking for a second, possibly more geometric proof. It is led by the following ideas: The unit sphere in the tangent space of L/G at eG can be identified with the boundary at infinity of L/G in a G -equivariant way by mapping v to $\gamma_v(\infty)$, where γ_v is the geodesic in L/G starting at eG in direction v . A symmetric R -space G/G_ξ can thus be identified with an orbit $G(v)$ of the linear isotropy representation (s -representation) of L/G and inherits a Riemannian metric. This is called the *standard embedding* of the symmetric R -space G/G_ξ . FERUS [F] has shown that a standardly embedded symmetric R -space is an *extrinsically symmetric space* (that is a submanifold of a Euclidean space which is invariant under the reflections along all its affine normal spaces) and, more importantly, that any compact extrinsically symmetric space of positive dimension is congruent to a

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symmetric R -space considered as an s -orbit (see [EH] for a short proof). Now our main result can be formulated as follows:

THEOREM 1.1. *A Riemannian symmetric space X with rectangular unit lattice admits a full isometric embedding Φ in a Euclidean space E whose image is extrinsically symmetric. Moreover, such an embedding is unique up to congruence.*

By means of FERUS' result the first part of Theorem 1.1 shows that a Riemannian symmetric space with rectangular unit lattice is affinely equivalent to a symmetric R -space and thus provides a new proof of [L3, Satz 6]. As a direct consequence of the uniqueness statement of Theorem 1.1 every (intrinsic) isometry between full compact extrinsically symmetric spaces is induced from an isometry between the ambient Euclidean spaces (Corollary 4.5).

Note that the (easier) converse of the first part of Theorem 1.1 is also true: The unit lattice of a compact extrinsically symmetric space is rectangular (cf. Theorem A.6). We refer to [EHQ] for a geometric proof (see also [EQT1]). In the context of symmetric R -spaces this statement is essentially [L3, Satz 5] and can already be found in TAKEUCHI [T1, Cor. 1, p. 158f. and Cor. 1, p. 165].

Our proof of Theorem 1.1 is based on the following considerations: Let $X = G/K$ be a compact Riemannian symmetric space with rectangular unit lattice. Since compact extrinsically symmetric spaces are essentially s -orbits that are symmetric R -spaces, we are looking for a G -equivariant embedding $\Phi : X \rightarrow E$ in a Euclidean vector space E on which G acts by orthogonal transformations. Such an embedding Φ maps eK to a non-zero element $v_o \in E$ which is fixed by K . An irreducible representation of G with a non-zero K -fixed vector is called K -spherical and may be seen as an irreducible representation of X . According to a theorem of CARTAN, HELGASON and TAKEUCHI (see [T3] and Appendix A.2) the complex irreducible representations of $X = G/K$ are parameterized by orbits of the Weyl group W_X of X acting on the dual Γ_X^* of the unit lattice Γ_X of X , in complete analogy to the highest weight theorem for representations of compact Lie groups. If X is indecomposable (that is not a Riemannian product of two symmetric spaces of positive dimension) and $\varepsilon_1, \dots, \varepsilon_r \in \Gamma_X^*$ is the dual basis of an orthogonal basis of Γ_X , then we show that $W_X(\varepsilon_1)$ contains $\{\varepsilon_1, \dots, \varepsilon_r\}$ (Proposition 3.1). It is therefore tempting to consider the complex irreducible K -spherical G -module V with the lowest possible 'highest weight' $W_X(\varepsilon_1)$. If $v_0 \in V$ is a non-zero fixed vector of K and V is endowed with the real part of a suitable G -invariant hermitian metric, then $\Phi : X = G/K \rightarrow V$, $gK \mapsto gv_0$, turns out to be an isometric embedding. Moreover, since there are no non-zero weights strictly lower than ε_1 , Φ maps maximal tori of X to Clifford tori in V , that is to products of circles in pairwise orthogonal planes. By a previous result of the authors [EHQ] (see Theorem A.6) $\Phi(X)$ is extrinsically symmetric.

To prepare the proof as well as to investigate Riemannian symmetric spaces with rectangular unit lattice by themselves we study in the first part of this paper the relationship between the unit lattice and the (restricted) root system of a compact Riemannian symmetric space. Several of the results are known. However, our proofs are more direct neither using the correspondence between symmetric spaces with rectangular unit lattice and symmetric R -spaces nor classification results. We formalize the relationship between

unit lattices and root systems by defining the notion of a *Euclidean root datum* of a compact Riemannian symmetric space generalizing the root data of compact Lie groups (equivalently of complex reductive algebraic groups). A Euclidean root datum not only contains the information about the root system, but also the position of the symmetric space within the poset of symmetric quotients of its universal cover and, in particular, its fundamental group. We compute explicitly the Euclidean root data for indecomposable symmetric spaces with rectangular unit lattice and show that there are precisely five possibilities (Theorem 3.6). As applications we get a generalization of the polysphere theorem for hermitian symmetric spaces of compact type to all simply connected symmetric spaces with rectangular unit lattice (Theorem 3.8) and an explicit formula for the eigenvalues of the Laplacian for this class of spaces (Proposition 5.2). The last result implies in particular that the extrinsically symmetric embeddings Φ from Theorem 1.1 are embeddings in the first eigenspace of the Laplacian in almost all cases (Corollary 5.3), confirming a result of OHNITA [O] about standard embeddings of symmetric R -spaces.

2. Preliminaries

We recall some standard facts about symmetric spaces and fix notations. For more details see e. g. [H3, L1].

In the following a symmetric space will always mean a connected Riemannian symmetric space of positive dimension, usually denoted by X . Thus for each $x \in X$ there exists an involutive isometry s_x of X , called the *geodesic symmetry* at x , that has x as an isolated fixed point. We denote by $I(X)$ the group of isometries of X and by $G = G(X)$ the closed subgroup of $I(X)$ that is generated by the transvections $s_p \circ s_q$ with $p, q \in X$. Then G is a connected Lie group that acts transitively and effectively on X and we call it the *transvection group* of X . If X is compact, then G coincides with the connected component of $I(X)$ containing the identity.

We now fix a base point $p \in X$ and let $K = G_p$ be the isotropy group of G at p . This yields an identification $G/K \rightarrow X$, $gK \mapsto g.p := g(p)$, which becomes an isometry if G/K is endowed with the pull-back metric. This is G -invariant and the geodesic symmetry s_p of X at p corresponds on G/K to the map $gK \mapsto \sigma(g)K$, where σ is the involutive automorphism of G given by the conjugation with s_p . Since K is an open subgroup of $G^\sigma = \{g \in G : \sigma(g) = g\}$, (G, K) is a symmetric pair. The differential of σ at $e \in G$, also denoted by σ , induces a splitting of the Lie algebra $\mathfrak{g}$ of G as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into the fixed point sets $\mathfrak{k}$ and $\mathfrak{p}$ of σ and $-\sigma$, respectively. Note that $\mathfrak{k}$ is the Lie algebra of K . We identify $\mathfrak{p}$ with $T_p X$ via $v \mapsto \frac{d}{dt}|_{t=0} (\exp(tv).p)$, where $\exp : \mathfrak{g} \rightarrow G$ is the Lie theoretic exponential map, and we take on $\mathfrak{p}$ the pull-back inner product. Using this identification the curvature tensor R of X at p is given by $R(X, Y)Z = [Z, [X, Y]]$ for all $X, Y, Z \in \mathfrak{p}$ and the Riemannian exponential map $\text{Exp}_p : T_p X \rightarrow X$ of X at p satisfies $\text{Exp}_p(V) = \exp(V).p$ for all $V \in \mathfrak{p}$.

We further fix a maximal flat F_X of X through p , that is a connected, flat, totally geodesic submanifold of X of maximal dimension. Since G acts transitively on the set of pairs (x, F) , where $x \in X$ and F is a maximal flat of X containing x , the *rank* of X , which is the dimension of a maximal flat of X , is well-defined. The tangent space of F_X at p is identified with a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Conversely, any maximal

abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ gives rise to the maximal flat $\exp(\mathfrak{a}).p$ of X through p .

The *unit lattice* of a symmetric space X (with respect to (p, F_X)) is

$$\Gamma_X := \{H \in \mathfrak{a} : \exp(H).p = p\} = \{H \in \mathfrak{a} : \exp(H) \in K\},$$

which is a *lattice* in $\mathfrak{a}$, that is a discrete subgroup of the additive group $\mathfrak{a}$. Γ_X is full in $\mathfrak{a}$, that is it spans $\mathfrak{a}$, if and only if X is compact. In any case the Riemannian exponential map identifies $\mathfrak{a}/\Gamma_X$ with F_X . If X is compact $\mathfrak{a}/\Gamma_X$ is a flat torus and F_X is called a *maximal torus* of X , also denoted by T_X .

Assume from now on that X is compact or, more generally, covers a compact symmetric space. For an element $\alpha \in \mathfrak{a}^* = \text{Hom}(\mathfrak{a}, \mathbb{R})$ let

$$\mathfrak{g}_\alpha := \{Z \in \mathfrak{g} \otimes \mathbb{C} : \text{ad}(H)Z = 2\pi i\alpha(H)Z \text{ for all } H \in \mathfrak{a}\}$$

and

$$\mathcal{R}_X := \{\alpha \in \mathfrak{a}^* \setminus \{0\} : \mathfrak{g}_\alpha \neq \{0\}\}.$$

Then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ for all $\alpha, \beta \in \mathfrak{a}^*$ and

$$\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \mathcal{R}_X} \mathfrak{g}_\alpha.$$

$\mathcal{R}_X$ is called the *root system* of X (with respect to (p, F_X)) although it is only a root system in a subspace of $\mathfrak{a}^*$. Note that up to isomorphism the triple $(\mathfrak{a}, \Gamma_X, \mathcal{R}_X)$ only depends on X , that is if (p, F) is replaced by (p', F') and thus $\mathfrak{a}$, Γ_X , $\mathcal{R}_X$ by $\mathfrak{a}'$, Γ'_X , $\mathcal{R}'_X$ then there exists a linear isometry $\varphi : \mathfrak{a} \rightarrow \mathfrak{a}'$ with $\varphi(\Gamma_X) = \Gamma'_X$ and $\varphi(\mathcal{R}_X) = \mathcal{R}'_X$ where $\varphi(\alpha) = \alpha \circ \varphi^{-1}$ for $\alpha \in \mathfrak{a}^*$. The interplay between the unit lattice and the set of roots is discussed in Appendix A.1.

If $\pi : X \rightarrow X'$ is a Riemannian covering of symmetric spaces then $G(X)$ covers $G(X')$ as the geodesic symmetries of X push down to geodesic symmetries of X' inducing a surjective homomorphism $G = G(X) \rightarrow G(X') = G'$. Its kernel is contained in the group of deck transformations of π and thus discrete. We may assume $\mathfrak{g} = \mathfrak{g}'$ for the Lie algebras of G and G' and further $\mathfrak{k} = \mathfrak{k}'$ and $\mathfrak{a} = \mathfrak{a}'$ by choosing appropriate base points and maximal flats in X and X' . In particular $\Gamma_X \subset \Gamma_{X'}$ and $\mathcal{R}_X = \mathcal{R}_{X'}$. Since the universal cover of X splits as $X_0 \times X_1$, where X_0 is a Euclidean space and X_1 is a simply connected symmetric space of compact type (one of them possibly a point), $\mathfrak{a}$ splits orthogonally as $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1$ with $\mathfrak{a}_0 = \bigcap_{\alpha \in \mathcal{R}_X} \ker(\alpha)$. Restriction to $\mathfrak{a}_1$ yields an isometric isomorphism from $\text{span}(\mathcal{R}_X)$ to $\mathfrak{a}_1^*$ that maps $\mathcal{R}_X$ onto the root system of X_1 . In particular, $\mathcal{R}_X$ is a root system in its linear span.

The *Weyl group* W_X of X (corresponding to (p, F)) is the subgroup of the orthogonal group of $\mathfrak{a}$ that is generated by the orthogonal reflections s_α along the hyperplanes $\ker(\alpha)$ with $\alpha \in \mathcal{R}_X$. Recall that

$$s_\alpha(H) = H - \alpha(H)\alpha^\vee$$

for all $H \in \mathfrak{a}$, where $\alpha^\vee \in \mathfrak{a}$ is the unique vector orthogonal to $\ker(\alpha)$ with $\alpha(\alpha^\vee) = 2$. It is equal to $\frac{2}{\|H_\alpha\|^2} H_\alpha$, where $H_\alpha \in \mathfrak{a}$ is determined by $\alpha(H) = \langle H_\alpha, H \rangle$ for all $H \in \mathfrak{a}$. $\mathcal{R}_X^\vee := \{\alpha^\vee : \alpha \in \mathcal{R}_X\}$ is called the *set of inverse roots* and is a root system in $\mathfrak{a}_0^\perp$. The Weyl group has a second description as $W_X = M'/M$ with $M' = \{k \in K : \text{Ad}(k)\mathfrak{a} = \mathfrak{a}\}$ and $M = \{k \in K : \text{Ad}(k)H = H \text{ for all } H \in \mathfrak{a}\}$ (see e.g. [L1, Prop. 2.2, p. 67]). In particular, W_X preserves the unit lattice Γ_X and the set of roots $\mathcal{R}_X$. Note that W_X also acts on $\mathfrak{a}^*$ by $w.\varphi := \varphi \circ w^{-1}$ for all $w \in W_X$ and $\varphi \in \mathfrak{a}^*$.

A symmetric space X is called *decomposable* if X is isometric to a Riemannian product of positive dimensional Riemannian manifolds. Otherwise X is called *indecomposable*.

LEMMA 2.1 (Splitting Lemma for compact symmetric spaces). *Let $X \cong G/K$ be a compact symmetric space. Then the following assertions are equivalent:*

- (1) X is decomposable.
- (2) $\mathfrak{a}$ splits as an orthogonal direct sum $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ of non-trivial W_X -invariant subspaces such that the unit lattice Γ_X of X splits accordingly as $\Gamma_X = (\Gamma_X \cap \mathfrak{a}_1) + (\Gamma_X \cap \mathfrak{a}_2)$.

PROOF. Obviously the first statement implies the second one. We now assume the second assertion. Since for all $\alpha \in \mathcal{R}_X$ the reflection s_α is an element of W_X , $\mathfrak{a}_1$ and $\mathfrak{a}_2$ split into eigenspaces of s_α . The (-1) -eigenspace of such an s_α is generated by $H_\alpha \in \mathfrak{a}$. Thus for any $\alpha \in \mathcal{R}_X$ we have either $H_\alpha \in \mathfrak{a}_1$ or $H_\alpha \in \mathfrak{a}_2$. Hence the set of roots of X is a disjoint union $\mathcal{R}_X = \mathcal{R}_1 \cup \mathcal{R}_2$ where $\mathcal{R}_j = \{\alpha \in \mathcal{R}_X : H_\alpha \in \mathfrak{a}_j\}$, $j = 1, 2$ (cf. [S, Prop. 10, p. V-21]). Thus $\mathfrak{p} \cong T_o X$ splits orthogonally into two curvature invariant subspaces $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$, where $\mathfrak{p}_j = \mathfrak{a}_j + \sum_{\alpha \in \mathcal{R}_j} (\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}) \cap \mathfrak{p}$, $j = 1, 2$. Consequently X splits locally as $X_1 \times X_2$, where $X_j = \exp(\mathfrak{p}_j).p$, $j \in \{1, 2\}$, are totally geodesic submanifolds of X (see e.g. [H3, Chap. IV, §7]), and X is covered by $X_1 \times X_2$.

On the other hand, the maximal torus of X splits globally as $T_X = \exp(\mathfrak{a}).p \cong \mathfrak{a}/\Gamma_X = (\mathfrak{a}_1 + \mathfrak{a}_2)/((\Gamma_X \cap \mathfrak{a}_1) + (\Gamma_X \cap \mathfrak{a}_2)) \cong (\mathfrak{a}_1/(\Gamma_X \cap \mathfrak{a}_1)) \times (\mathfrak{a}_2/(\Gamma_X \cap \mathfrak{a}_2)) = T_1 \times T_2$ where $T_j = \exp(\mathfrak{a}_j).p \cong \mathfrak{a}_j/(\Gamma_X \cap \mathfrak{a}_j)$ for $j = 1, 2$. Thus when restricted to $T_X = T_1 \times T_2$ (or to any other maximal torus of $X_1 \times X_2$) the covering $\pi : X_1 \times X_2 \rightarrow X$ is one-to-one. Let now q and q' be two points in $X_1 \times X_2$ with $\pi(q) = \pi(q')$. Since q and q' can be joint by a geodesic, there is a maximal torus of $X_1 \times X_2$ containing both points q and q' . Since the restriction of π to this maximal torus is injective, we get $q = q'$. Thus π is globally one-to-one and X splits as a product $X_1 \times X_2$. $\square$

3. The Roots of a Symmetric Space with Rectangular Unit Lattice

In this section we analyse the interplay between the unit lattice and the set of roots of symmetric spaces with rectangular unit lattice.

A symmetric space X has *rectangular* unit lattice $\Gamma_X \subset \mathfrak{a}$, if there exists an orthogonal basis $e_1, \dots, e_r$ of $\mathfrak{a}$ whose $\mathbb{Z}$ -span is Γ_X . If $e_1, \dots, e_r$ can be chosen to have the same length, then X is said to have *cubic* unit lattice. A symmetric space with rectangular unit lattice is necessarily compact as its maximal flats are compact.

Symmetric spaces with rectangular unit lattice include extrinsically symmetric space (see [EHQ] and the references therein), in particular hermitian symmetric spaces of

compact type. We will see in Section 4 that every symmetric space with a rectangular unit lattice can be realized as an extrinsically symmetric space.

In the remainder of this section X will always denote an indecomposable symmetric space of rank $r \geq 1$ with rectangular unit lattice Γ_X , $e_1, \dots, e_r$ an orthogonal basis of Γ_X , and $\varepsilon_1, \dots, \varepsilon_r$ the corresponding dual basis of $\mathfrak{a}^*$.

PROPOSITION 3.1. *The Weyl group W_X leaves $\{\pm e_1, \dots, \pm e_r\}$ invariant and there are two possibilities:*

- (I) *X is of compact type and W_X acts transitively on $\{\pm e_1, \dots, \pm e_r\}$.*
- (II) *The Euclidean factor in the universal cover of X is 1-dimensional and the action of W_X on $\{\pm e_1, \dots, \pm e_r\}$ has two different orbits which differ by a sign. Each orbit is a basis of Γ_X and has the form $\{\lambda_1 e_1, \dots, \lambda_r e_r\}$ for suitable $\lambda_j \in \{-1, 1\}$.*

In particular, $e_1, \dots, e_r$ is always a cubic basis.

PROOF. By orthogonality the set of shortest vectors in $\Gamma_X \setminus \{0\}$ is contained in the set $\{\pm e_1, \dots, \pm e_r\}$. Thus $\{e_1, \dots, e_r\}$ contains an element of shortest length in $\Gamma_X \setminus \{0\}$, say e_1 . Since W_X acts isometrically on $\mathfrak{a}$ leaving Γ_X invariant, $W_X(e_1) \cup W_X(-e_1)$ consists of shortest elements of $\Gamma_X \setminus \{0\}$ and is therefore contained in $\{\pm e_1, \dots, \pm e_r\}$. Let $J = \{j \in \{1, \dots, r\} : e_j \in W_X(e_1) \cup W_X(-e_1)\}$. Then $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ with $\mathfrak{a}_1 := \text{span}_{\mathbb{R}}\{e_j : j \in J\}$ and $\mathfrak{a}_2 = \text{span}_{\mathbb{R}}\{e_k : k \in \{1, \dots, r\} \setminus J\}$. This is a W_X -invariant orthogonal decomposition of $\mathfrak{a}$ such that $\Gamma_X = (\Gamma_X \cap \mathfrak{a}_1) + (\Gamma_X \cap \mathfrak{a}_2)$. Since X is indecomposable, Lemma 2.1 yields $J = \{1, \dots, r\}$. This shows $W_X(e_j) \cup W_X(-e_j) = \{\pm e_1, \dots, \pm e_r\}$ for every $j \in \{1, \dots, r\}$. Thus the action of W_X on $\{\pm e_1, \dots, \pm e_r\}$ has at most two orbits. If there are two different orbits, then each orbit must be a basis of Γ_X and the two orbits differ by a sign.

A positive dimensional Euclidean factor in the de Rham decomposition for the universal cover of X corresponds to a nonzero linear subspace $\mathfrak{a}_0$ in $\mathfrak{a}$ on which W_X acts trivially. If W_X acts transitively on $\{\pm e_1, \dots, \pm e_r\}$, then the only fixed vector of W_X in $\mathfrak{a}$ is 0, that is $\mathfrak{a}_0 = \{0\}$. In the other case we may assume (after suitably changing the signs) that $\{e_1, \dots, e_r\}$ is a W_X -orbit. Then $\mathfrak{a}_0 = \mathbb{R} \cdot \sum_{j=1}^r e_j$ and the Euclidean factor in the de Rham decomposition of $\tilde{X}$, the universal cover of X , has dimension one. $\square$

PROPOSITION 3.2 (c. f. [L3, Lemma 7]). $2\mathcal{R}_X \subset \{\pm \varepsilon_j, \pm 2\varepsilon_j, \pm(\varepsilon_j \pm \varepsilon_k) : 1 \leq j, k \leq r\}$.

PROOF. Let $\alpha \in \mathcal{R}_X$. Then $2\alpha = \sum_{j=1}^r m_j \varepsilon_j$ for some $m_j \in \mathbb{Z}$ by (i) in Theorem A.1. Thus $\frac{1}{2}\alpha^\vee = \frac{2}{\sum_{j=1}^r m_j^2} \sum_{k=1}^r m_k e_k$. By (ii) in Theorem A.1 we get $\frac{2m_k}{\sum_{j=1}^r m_j^2} \in \mathbb{Z}$ for all $k = 1, \dots, r$ and the claim follows. $\square$

PROPOSITION 3.3. *After suitable changes of signs for some of the ε_i we have*

$$\{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq r\} \subset 2\mathcal{R}_X.$$

PROOF. Let $I = \{1, \dots, r\}$. We show by induction on k that for all $k \in I$ we have:

$(*)_k$ There exists $I' \subset I$ with $|I'| = k$ and $\varepsilon_i - \varepsilon_j \in 2\mathcal{R}_X$ for all $i, j \in I'$, $i \neq j$, after a possible change of signs of some of the ε_i , $i \in I'$.

While $(*)_r$ proves the Proposition, $(*)_1$ is obvious.

Assume that I' satisfies $(*)_k$ for some $k < r$. Then $\{\pm\varepsilon_i : i \in I'\}$ is not W_X -invariant according to Proposition 3.1. More precisely, there exists $\ell \in I'$ and $m \in I \setminus I'$ with $\varepsilon_\ell + \varepsilon_m \in 2\mathcal{R}_X$ or $\varepsilon_\ell - \varepsilon_m \in 2\mathcal{R}_X$ as W_X is generated according to Proposition 3.2 by reflections of the form s_{ε_i} , $s_{\varepsilon_i + \varepsilon_j}$ or $s_{\varepsilon_i - \varepsilon_j}$, $i, j \in I$, $i \neq j$. Note that these generators only occur if the corresponding roots occur and that they leave $\{\pm\varepsilon_n\}$ invariant for all $n \in I$ except in case of $s_{\varepsilon_i \pm \varepsilon_j}$ which both interchange $\{\pm\varepsilon_i\}$ with $\{\pm\varepsilon_j\}$. After a possible change of sign of ε_m we may assume $\varepsilon_\ell - \varepsilon_m \in 2\mathcal{R}_X$. But then also $\varepsilon_i - \varepsilon_m = s_{\varepsilon_i - \varepsilon_\ell}(\varepsilon_\ell - \varepsilon_m) \in 2\mathcal{R}_X$ for all $i \in I'$. Thus $I' \cup \{m\}$ satisfies $(*)_{k+1}$ completing the induction step. $\square$

PROPOSITION 3.4. $\mathcal{R}_X$ is one of the following classical root systems:

$$\begin{aligned} \mathcal{R}_1 &= \frac{1}{2}\{\varepsilon_j - \varepsilon_k \mid 1 \leq j \neq k \leq r\} & (A_{r-1}), \\ \mathcal{R}_2 &= \frac{1}{2}\{\pm\varepsilon_j \mid 1 \leq j \leq r\}, \pm\varepsilon_j \pm \varepsilon_k \mid 1 \leq j < k \leq r\} & (B_r), \\ \mathcal{R}_3 &= \frac{1}{2}\{\pm 2\varepsilon_j \mid 1 \leq j \leq r\}, \pm\varepsilon_j \pm \varepsilon_k \mid 1 \leq j < k \leq r\} & (C_r), \\ \mathcal{R}_4 &= \frac{1}{2}\{\pm\varepsilon_j \pm \varepsilon_k \mid 1 \leq j < k \leq r\} & (D_r), \\ \mathcal{R}_5 &= \frac{1}{2}\{\pm\varepsilon_j \mid 1 \leq j \leq r\}, \pm 2\varepsilon_j \mid 1 \leq j \leq r\}, \pm\varepsilon_j \pm \varepsilon_k \mid 1 \leq j < k \leq r\} & (BC_r). \end{aligned}$$

Except for the first case $\mathcal{R}_X$ is a root systems in $\mathfrak{a}^*$ and X is of compact type and $W_X.e_1 = \{\pm e_1, \dots, \pm e_r\}$. In the first case $\mathcal{R}_X$ is a root system in the codimension-one subspace $\{\alpha \in \mathfrak{a}^* : \alpha(\sum_{j=1}^n e_j) = 0\}$ of $\mathfrak{a}^*$ and X splits off locally an S^1 -factor. Moreover, $W_X.e_1 = \{e_1, \dots, e_r\}$ in the first case. Except for D_2 , $\mathcal{R}_X$ is an irreducible root system in its span.

PROOF. By Proposition 3.3 we may assume that $\varepsilon_j - \varepsilon_k \in 2\mathcal{R}_X$ for all $j, k \in I = \{1, \dots, r\}$ with $j \neq k$. Now we distinguish two cases:

- (i) There exists $j \in I$ with $\{\varepsilon_j, 2\varepsilon_j\} \cap 2\mathcal{R}_X \neq \emptyset$:
If $\varepsilon_j \in 2\mathcal{R}_X$ or $2\varepsilon_j \in 2\mathcal{R}_X$ for one (and hence all) $j \in I$ then the reflections $s_{\varepsilon_j} \in W_X$ allow us to change the signs of the ε_j individually. Thus all $\varepsilon_j \pm \varepsilon_k \in 2\mathcal{R}_X$. This yields the root systems of type B_r, C_r and BC_r for $r \geq 1$ depending on whether ε_j or $2\varepsilon_j$ or both are contained in $2\mathcal{R}_X$.
- (ii) For all $j \in I$ we have $\{\varepsilon_j, 2\varepsilon_j\} \cap 2\mathcal{R}_X = \emptyset$:
Then either $2\mathcal{R}_X = A_{r-1} = \{\pm(\varepsilon_j - \varepsilon_k) : j, k \in I, j \neq k\}$ or some (hence all) $\varepsilon_j + \varepsilon_k \in 2\mathcal{R}_X$ and $2\mathcal{R}_X = D_r = \{\pm(\varepsilon_j \pm \varepsilon_k) : j, k \in I, j \neq k\}$.

$\square$

NOTE 3.5. The root systems of irreducible hermitian symmetric spaces of compact type (C_r and BC_r) are determined in [M, Thm. 2, p. 362]. A list of root systems

(and fundamental groups) of irreducible symmetric R -spaces can be found in [T2, p. 305] which is based on an extensive study of symmetric R -spaces in [T1].

A *Euclidean root datum* is a triple $(V, \Gamma, \mathcal{R})$ consisting of a Euclidean vector space V , a full lattice Γ in V and a root system $\mathcal{R}$ in a subspace of V^* such that $\Gamma_0(\mathcal{R}) \subset \Gamma \subset \Gamma_1(\mathcal{R})$ where $2\Gamma_0(\mathcal{R})$ is the $\mathbb{Z}$ -span of $\mathcal{R}^\vee$ and $2\Gamma_1(\mathcal{R}) = \{v \in V : \alpha(v) \in \mathbb{Z} \text{ for all } \alpha \in \mathcal{R}\}$. To each Riemannian symmetric space Y one can associate a Euclidean root datum (up to isomorphism) by taking $V = \mathfrak{a}$, $\Gamma = \Gamma_Y$ and $\mathcal{R} = \mathcal{R}_Y$ (see Appendix A.1).

Let $(V_r, \Gamma_L, \mathcal{R}_j)$, $j = 1, \dots, 5$, be the Euclidean root datum with V_r an r -dimensional Euclidean vector space, Γ_L a lattice in V_r spanned by an orthogonal basis $e_1, \dots, e_r$ of common length $L > 0$ and $\mathcal{R}_j$ one of the five root systems explicitly described in Proposition 3.4 in terms of the dual basis $\varepsilon_1, \dots, \varepsilon_r$ of $e_1, \dots, e_r$. We say that X is of type $\widehat{A}_{r-1}$, $\widehat{B}_r$, $\widehat{C}_r$, $\widehat{D}_r$ or $\widehat{BC}_r$ if its Euclidean root datum is up to scaling (that is up to a choice of L) isomorphic to $(V_r, \Gamma_L, \mathcal{R}_j)$ with $\mathcal{R}_j$ as in Proposition 3.4 and of the corresponding type.

THEOREM 3.6. *X is of type $\widehat{A}_{r-1}$, $\widehat{B}_r$, $\widehat{C}_r$, $\widehat{D}_r$ or $\widehat{BC}_r$ with $r \geq 2$ in case $\widehat{D}_r$ and $r \geq 1$ otherwise. The type of X is unique and each type occurs. Moreover,*

$$\pi_1(X) \cong \begin{cases} \{1\} & \text{if } X \text{ is of type } \widehat{C}_r \text{ or } \widehat{BC}_r \\ \mathbb{Z}_2 & \text{if } X \text{ is of type } \widehat{B}_r \text{ or } \widehat{D}_r \\ \mathbb{Z} & \text{if } X \text{ is of type } \widehat{A}_{r-1} \end{cases}.$$

PROOF. It follows from Propositions 3.1 and 3.4 that the Euclidean root datum of X is isomorphic to $(V_r, \Gamma_L, \mathcal{R}_j)$ for some $L > 0$ and some $j = 1, \dots, 5$. Thus the type of X is one of the above. Note that the types $\widehat{A}_0$ and $\widehat{D}_1$ coincide as $\mathcal{R}_j = \emptyset$ in both cases. The examples U_r , SO_{2r+1} , Sp_r , SO_{2r} and $G_r(\mathbb{C}^{2r+1})$ show that all types occur. Since $\pi_1(X) \cong \Gamma_X/\Gamma_0$ by Theorem A.1, the statement on the fundamental groups follows by a simple calculation. Therefore the five type can be distinguished by their root systems and fundamental groups implying uniqueness. $\square$

REMARK 3.7. Let X be an indecomposable symmetric space with rectangular unit lattice. Then Case (II) occurs in Proposition 3.1 if and only if X is of type $\widehat{A}_{r-1}$.

Let Y be a symmetric space and let r be a positive integer. A submanifold S of Y is called *polysphere* of rank r , if S is totally geodesic and isometric to a Riemannian product of r round spheres of dimension at least 2.

THEOREM 3.8 (Polysphere theorem). *Let Y be a compact symmetric space of rank $r \geq 1$. Then the following statements are equivalent:*

- (i) Y contains a polysphere of rank r .
- (ii) Y is simply connected and has rectangular unit lattice.

PROOF. Assuming (i) let S be a polysphere of rank r in Y . Then the maximal torus T of S , which is a Riemannian product of circles, is also a maximal torus in Y . Thus

Y has rectangular unit lattice. Now every closed curve in Y is homotopic to a closed geodesic γ in T . Since S is simply connected (being a product of spheres of dimension at least 2), γ is contractible in S and hence in Y . Thus Y is simply connected.

If (ii) holds we may further assume that Y is indecomposable. Then Y is of type $\widehat{C}_r$ or $\widehat{BC}_r$ by Theorem 3.6. In both cases the roots $\varepsilon_1, \dots, \varepsilon_r$ are strongly orthogonal, that is $\varepsilon_j \pm \varepsilon_k \notin R_Y \cup \{0\}$ for $j \neq k$. Let $j \in \{1, \dots, r\}$. Since in both cases $2\varepsilon_j \notin R_Y$ we get a Lie subtriple $\mathfrak{s}_j = \mathbb{R} \cdot H_{\varepsilon_j} + (\mathfrak{g}_{\varepsilon_j} + \mathfrak{g}_{-\varepsilon_j}) \cap \mathfrak{p}$ of $\mathfrak{p}$. Indeed, $[\mathfrak{s}_j, \mathfrak{s}_j] \subset \mathfrak{k}_0 \oplus (\mathfrak{g}_{\varepsilon_j} + \mathfrak{g}_{-\varepsilon_j}) \cap \mathfrak{k}$, $\mathfrak{k}_0$ the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, and $[\mathfrak{g}_{\pm\varepsilon_j}, \mathfrak{g}_{\pm\varepsilon_j}]$ is orthogonal to all $H \in \mathfrak{a}$ with $H \perp H_{\varepsilon_j}$. The corresponding totally geodesic submanifold S_j of Y has non-vanishing constant sectional curvature and is therefore covered by a simply connected sphere. Strong orthogonality implies that $\mathfrak{s}_j$ and $\mathfrak{s}_k$ are perpendicular for $j \neq k$ and that $\mathfrak{s} = \bigoplus_{j=1}^r \mathfrak{s}_j$ is again a Lie subtriple of $\mathfrak{p}$. The totally geodesic submanifold S of Y corresponding to $\mathfrak{s}$ is covered by a product of r simply connected spheres. The root system $\mathcal{R}_S = \{\pm\varepsilon_1, \dots, \pm\varepsilon_r\}$ of S is a root subsystem of $\mathcal{R}_Y$ and the unit lattice of S is $\Gamma_S = \text{span}_{\mathbb{Z}}\{e_1, \dots, e_r\} = \text{span}_{\mathbb{Z}}\{\frac{1}{2}\varepsilon_1^\vee, \dots, \frac{1}{2}\varepsilon_r^\vee\}$. By Theorem A.1 S is simply connected and hence a polysphere of rank r . $\square$

REMARK 3.9. We see slightly more: The polyspheres in Theorem 3.8 are actually ‘poly-Helgason-spheres’, that is they are products of maximal dimensional totally geodesic spheres whose sectional curvature realizes the maximum of the sectional curvatures of Y (see [H1]).

Theorem 3.8 generalizes the ‘polysphere theorem’ for hermitian symmetric spaces of compact type (see e.g. [W, p. 280]). Note that not all simply connected Riemannian symmetric spaces with rectangular unit lattice are hermitian, e. g. the compact symplectic groups Sp_n , $n \geq 2$, the quaternionic Grassmannians or the Cayley plane.

4. Extrinsically Symmetric Embeddings

In this section we show Theorem 1.1. As explained in the introduction, it is closely related to a result of LOOS in [L3], but our arguments and our methods are entirely different. Let X be a Riemannian symmetric space with rectangular unit lattice.

PROOF OF THE EXISTENCE STATEMENT IN THEOREM 1.1. We may assume that X is indecomposable and has positive dimension. By Proposition 3.1 we can choose an orthogonal basis $B = \{e_1, \dots, e_r\}$ of Γ_X such that

(I) $W_X(e_1) = B \cup (-B)$ if X is of compact type, or

(II) $W_X(e_1) = B$ if X splits off a local S^1 -factor.

We choose a Weyl chamber $\mathfrak{a}_+ \subset \mathfrak{a}$ of X containing e_1 in its closure. As before we extend $\mathfrak{a}$ to a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and choose a Weyl chamber in $\mathfrak{t}$ containing $\mathfrak{a}_+$ in its closure. This induces a partial ordering on $\mathfrak{t}^*$. Let $\{\varepsilon_1, \dots, \varepsilon_r\} \subset \mathfrak{a}^* \subset \mathfrak{t}^*$ be the basis dual to B , that is $\varepsilon_j(e_k) = 0$ if $j \neq k$ and $\varepsilon_j(e_j) = 1$.

By the Theorem of Cartan–Helgason–Takeuchi, Theorem A.3, there exists a K -spherical G -module V with highest weight ε_1 , which is unique up to equivalence. We

now choose an element $v^o \in V^K \setminus \{0\}$. This choice is unique up to a complex factor, since V^K has dimension one. We write v^o according to the weight space decomposition of V (see Equation (8)) as

$$v^o = \sum_{\omega \in \Omega_V} v_\omega^o = \sum_{\mu \in M} v_\mu^o \quad (1)$$

with $v_\mu^o \in V_\mu$ and $M = \{\mu \in \Omega_V : v_\mu^o \neq 0\}$, where Ω_V is the set of weights for V (see Appendix A.2). From $K.v^o = \{v^o\}$ we see that M is W_X -invariant and that $(K \cap \exp(\mathfrak{t})).v_\mu^o = v_\mu^o$ and therefore $\mu(\mathfrak{t} \cap \mathfrak{k}) = 0$ and $\mu(\Gamma_X) \subset \mathbb{Z}$ for all $\mu \in M$. This shows that $M \subset \Gamma_X^* = \text{span}_{\mathbb{Z}}(\varepsilon_1, \dots, \varepsilon_r)$. Since ε_1 is the highest weight of V , we get $M \setminus \{0\} \subset \{\pm\varepsilon_1, \dots, \pm\varepsilon_r\}$. Now $\varepsilon_1 \in M$, since the projection of the highest weight space V_{ε_1} onto V^K is non-zero (see [T3, Lem. 3, p. 92]). As $M \setminus \{0\}$ is W_X -invariant we have

$$\begin{aligned} M \setminus \{0\} &= \{\pm\varepsilon_1, \dots, \pm\varepsilon_r\} \text{ in Case (I) } \text{ or} \\ M \setminus \{0\} &= \{\varepsilon_1, \dots, \varepsilon_r\} \text{ in Case (II).} \end{aligned} \quad (2)$$

We now consider the G -equivariant (and therefore smooth) map

$$\Phi : X \rightarrow E := \text{span}_{\mathbb{R}}(G.v^o), \quad g.p \mapsto g.v^o. \quad (3)$$

Let $(\cdot, \cdot)_{\mathbb{C}}$ be a G -invariant hermitian inner product on V , then the G -invariant real inner product $\text{Re}((\cdot, \cdot)_{\mathbb{C}})$ makes E a real Euclidean G -module.

For any $H \in \mathfrak{a}$ we have

$$\Phi(\exp(H).p) = \exp(H).v^o = v_0^o + \sum_{\mu \in M \setminus \{0\}} e^{2\pi i \mu(H)} v_\mu^o \quad (4)$$

and by Equation (2)

$$\begin{aligned} \Phi(\exp(H).p) - v_0^o &= \sum_{j=1}^r (e^{2\pi i \varepsilon_j(H)} v_{\varepsilon_j}^o + e^{-2\pi i \varepsilon_j(H)} v_{-\varepsilon_j}^o) \text{ in Case (I) } \text{ or} \\ \Phi(\exp(H).p) - v_0^o &= \sum_{j=1}^r e^{2\pi i \varepsilon_j(H)} v_{\varepsilon_j}^o \text{ in Case (II).} \end{aligned}$$

By Example A.5 $\Phi(T_X)$ is a Clifford torus. The vectors $v_{\varepsilon_1}^o - v_{-\varepsilon_1}^o, \dots, v_{\varepsilon_r}^o - v_{-\varepsilon_r}^o$ (resp. $v_{\varepsilon_1}^o, \dots, v_{\varepsilon_r}^o$) have the same length as $\varepsilon_1, \dots, \varepsilon_r$ lie in the same W_X -orbit. Thus the unit lattice of $\Phi(T_X)$ is cubic.

Moreover, $\Phi|_{T_X} : T_X \rightarrow \Phi(T_X)$ is an isometry after possibly replacing the inner product on E by a multiple: In fact, the differential of $\Phi|_{T_X}$ at p maps the cubic basis $\{e_1, \dots, e_r\}$ of Γ_X onto a cubic basis of the unit lattice of $\Phi(T_X)$. To show that $\Phi|_{T_X}$ is injective we assume that $\Phi(\exp(H).p) = \Phi(p)$ for some $H \in \mathfrak{a}$. Then $\varepsilon_j(H) \in \mathbb{Z}$ for all $j \in \{1, \dots, r\}$. Thus $H \in \Gamma_X$ and $\exp(H).p = p$.

Since Φ is equivariant, it maps any maximal torus of X isometrically onto a Clifford torus of E . Therefore Φ is an isometric embedding of X with the property that every geodesic of $\Phi(X)$ is contained in a totally geodesic Clifford torus. By Theorem A.6 $\Phi(X)$ is an extrinsically symmetric space in E .

Finally, the affine hull $\text{aff}(\Phi(X))$ of $\Phi(X)$ coincides with E . Otherwise $0 \notin \text{aff}(\Phi(X))$ and there would be a unique nonzero element w of minimal length in $\text{aff}(\Phi(X))$. Since $\text{aff}(\Phi(X))$ is G -invariant, $G.w = w$, a contradiction to irreducibility. $\square$

REMARK 4.1. Following CARTAN [Ca] we can embed V G -equivariantly into the space $C^\infty(X)$ of complex valued smooth functions on X by $h : V \rightarrow C^\infty(X)$, $v \mapsto h_v$, where $h_v : X \rightarrow \mathbb{C}$, $x \mapsto (v, \Phi(x))_{\mathbb{C}}$. The submodule $h(V)$ of $C^\infty(X)$ contains the K -invariant function h_{v° . After multiplying v° with a suitable constant, h_{v° is given by

$$\begin{aligned} h_{v^\circ}(\exp(H).p) &= a + \sum_{j=1}^r \left(e^{2\pi i \varepsilon_j(H)} + e^{-2\pi i \varepsilon_j(H)} \right) \\ &= a + 2 \sum_{j=1}^r \cos(2\pi \varepsilon_j(H)) && \text{in Case (I) or} \\ h_{v^\circ}(\exp(H).p) &= a + \sum_{j=1}^r e^{-2\pi i \varepsilon_j(H)} && \text{in Case (II),} \end{aligned}$$

for all $H \in \mathfrak{a}$ (see Equation (4)). Here a is a real positive constant such that h_{v° is perpendicular to the constant functions with respect to the L^2 -inner product, that is the integral of h_{v° over X vanishes. It can be shown that $a = 0$ in Case (II) since 0 is not a weight as it is not contained in the convex hull of the Weyl orbit of ε_1 .

To finish the proof of Theorem 1.1 we next show:

THEOREM 4.2 (Uniqueness statement of Theorem 1.1). *Let X be a symmetric space with rectangular unit lattice and let $\Phi : X \rightarrow E$ and $\tilde{\Phi} : X \rightarrow \tilde{E}$ be two full, isometric, G -equivariant embeddings of X into real G -modules whose images are extrinsically symmetric spaces. Then there exists a G -equivariant linear isometry $F : E \rightarrow \tilde{E}$ such that $F \circ \Phi = \tilde{\Phi}$.*

Moreover, if X is indecomposable, then E is irreducible and its complexification E^c (as a G -module) is equivalent to V with $V \cong \bar{V}$ if X is of compact type and to $V \oplus \bar{V}$ with $V \not\cong \bar{V}$ if X splits off a local S^1 -factor, where V is the irreducible complex G -module with highest weight ε_1 .

PROOF. By Proposition A.10 we may assume that X is indecomposable. By Theorem A.6 $\Phi(X)$ is a submanifold of Clifford type. Given a full, isometric, G -equivariant embedding $\Phi : X \rightarrow E$ in a (real) Euclidean G -module E , such that $\Phi(X)$ is of Clifford type, we recover the representation data of E as follows. With the chosen base point $p \in X$ we set $v^\circ := \Phi(p) \in E \setminus \{0\}$. Note that v° is K -invariant and $\Phi(X) = G.v^\circ$. Now $E^c = E \otimes \mathbb{C}$ is a complex G -module. As in Equation (1) we decompose v° into weight vectors of E^c as $v^\circ = \sum_{\mu \in M} v_\mu^\circ$ where v_μ° lies in the weight space in E^c corresponding to the weight $\mu \in \Omega_{E^c}$ of E^c and $M = \{\mu \in \Omega_{E^c} : v_\mu^\circ \neq 0\}$. Let $\mu \in M$, then for all $H \in \mathfrak{t}$ we have $\exp(H).v_\mu^\circ = \sum_{\mu \in M} e^{2\pi i \mu(H)} v_\mu^\circ$ for all $H \in \mathfrak{t}$. If $\exp(H) \in K$ then $\exp(H).v^\circ = v^\circ$. Hence $\exp(H).v_\mu^\circ = v_\mu^\circ$ and therefore $\mu(H) \in \mathbb{Z}$ for all $\mu \in M$. This implies $\mu(\mathfrak{t} \cap \mathfrak{k}) = 0$ and $\mu \in \Gamma_X^*$.

Since Φ is isometric and $\Phi(X)$ is of Clifford type, $\Phi(T_X)$ is a Clifford torus in E by

Theorem A.6. Let $\{e_1, \dots, e_r\}$ be an orthogonal basis of Γ such that the curves

$$c_j : \mathbb{R} \rightarrow E, \quad t \mapsto \Phi(\exp(te_j).p) = \sum_{\mu \in M} e^{2\pi i t \mu(e_j)} v_\mu^o,$$

$j = 1, \dots, r$, have smallest positive period 1 and parameterize the generating circles of $\Phi(T_X)$ through v^o . In particular $c'_j(0)$ is perpendicular to $c'_k(0)$ for $j \neq k$. By Proposition 3.1 we may assume that $W_X.e_1 = \{\pm e_1, \dots, \pm e_r\}$ if X is of compact type, or $W_X.e_1 = \{e_1, \dots, e_r\}$ if X splits off a local S^1 -factor. We fix again a Weyl chamber $\mathfrak{a}_+$ in $\mathfrak{a}$ containing e_1 in its closure, a Weyl chamber $\mathfrak{t}_+$ in $\mathfrak{t}$ containing $\mathfrak{a}_+$ in its closure and the partial ordering on $\mathfrak{t}^*$ induced by $\mathfrak{t}_+$. Let $j \in \{1, \dots, r\}$. Since c_j has smallest positive period 1 and since $c_j(\mathbb{R})$ is a planar circle, there are vectors $a_j, b_j, d_j \in E$ with $c_j(t) = d_j + \cos(2\pi t)a_j + \sin(2\pi t)b_j$ for all $t \in \mathbb{R}$. It follows that $-4\pi^2 c'_j(0) = c''_j(0)$ and hence $\mu(e_j) = \mu(e_j)^3$ for all $\mu \in M$. Thus for any $\mu \in M$ and for any $j \in \{1, \dots, r\}$ we have $\mu(e_j) \in \{-1, 0, 1\}$. We extend the inner product on E to a G -invariant Hermitian inner product on E^c . Then v_μ^o is perpendicular to $v_{\mu'}^o$ for $\mu, \mu' \in M$ with $\mu \neq \mu'$. Since different generating circles of $\Phi(T_X)$ through v^o lie in perpendicular planes, we have

$$0 = \langle c''_j(0), c''_k(0) \rangle = 16\pi^4 \cdot \sum_{\mu \in M} (\mu(e_j))^2 (\mu(e_k))^2 \|v_\mu^o\|^2.$$

If $k \in \{1, \dots, r\}$ and $\mu \in M$ with $\mu(e_k) \neq 0$ (i. e. $\mu(e_k) = \pm 1$) then $\mu(e_j) = 0$ for all $j \neq k$. Summing up we have

$$M \setminus \{0\} \subset \{\pm \varepsilon_1, \dots, \pm \varepsilon_r\},$$

where $\varepsilon_1, \dots, \varepsilon_r \in \mathfrak{a}^* \subset \mathfrak{t}^*$ is the dual basis of $e_1, \dots, e_r$. Note that the only possible elements in the intersection of M with the closure of $\mathfrak{a}_+^*$ are ε_1 and $\bar{\varepsilon}_1 := w_o(-\varepsilon_1)$, where w_o is the unique element in W_X with $w_o(-\mathfrak{a}_+) = \mathfrak{a}_+$.

Since $\Phi(X) = G.v^o$ is full in E , $\Phi(X)$ is not contained in any proper complex affine subspace of $E \otimes \mathbb{C}$. Let V' be an irreducible complex G -submodule of $E \otimes \mathbb{C}$. Then the orthogonal projection of v^o onto V' is nonzero. Otherwise $G.v^o$ would be contained in the orthogonal complement of V' . Thus V' is K -spherical. Let $\lambda' \in (\Gamma_X^*)_+$ be the highest weight of V' (see Theorem A.3), then $\lambda' \in (M \setminus \{0\}) \cap (\Gamma_X^*)_+ \subset \{\varepsilon_1, \bar{\varepsilon}_1\}$. This shows that $E \otimes \mathbb{C}$ is a direct sum of K -spherical G -modules equivalent to $V(\varepsilon_1)$ or $V(\bar{\varepsilon}_1)$ in the notation of Theorem A.3. Note that $V(\bar{\varepsilon}_1) = \overline{V(\varepsilon_1)}$, the conjugate representation of $V(\varepsilon_1)$. Since $\overline{E \otimes \mathbb{C}}$ is equivalent to $E \otimes \mathbb{C}$, both $V(\varepsilon_1)$ and $\overline{V(\varepsilon_1)}$ occur in $E \otimes \mathbb{C}$. But, by fullness of $\Phi(X)$, none of the equivalence classes of these modules appears twice in $E \otimes \mathbb{C}$. Indeed, assume for simplicity that $E \otimes \mathbb{C} = U \oplus U'$, where U and U' are isomorphic irreducible complex G -modules. Let $f : U \rightarrow U'$ be a G -equivariant isomorphism and let $v^o = u + u'$ with $u \in U \setminus \{0\}$ and $u' \in U' \setminus \{0\}$. Then $u' = zf(u)$ for some $z \in \mathbb{C} \setminus \{0\}$, since U'^K has dimension one. But then $\{u + zf(u) : u \in U\}$ is a proper G -submodule of $E \otimes \mathbb{C}$ containing $\Phi(X)$, contradicting the fullness of $\Phi(X)$.

Summing up we have seen that $E \otimes \mathbb{C} \cong V(\varepsilon_1)$ if $\varepsilon_1 = \bar{\varepsilon}_1$, that is if X is of compact type or $E \otimes \mathbb{C} \cong V(\varepsilon_1) \oplus \overline{V(\varepsilon_1)}$ if $\varepsilon_1 \neq \bar{\varepsilon}_1$, that is if X splits off a local S^1 -factor. Thus $E \otimes \mathbb{C}$ is (up to equivalence) uniquely determined by X . Since the realification of $E \otimes \mathbb{C}$

is equivalent to $E \oplus E$, X also determines E uniquely (up to equivalence). Moreover E must be irreducible, since otherwise $E \otimes \mathbb{C}$ would split into two non-trivial self-conjugate submodules. If $E \otimes \mathbb{C}$ is irreducible (that is if X is of compact type) then E is a real form of $E \otimes \mathbb{C}$ and therefore $\dim_{\mathbb{R}}(E^K) = \dim_{\mathbb{C}}((E \otimes \mathbb{C})^K) = 1$. If X splits off a local S^1 -factor, then E is equivalent to the realification of $V(\varepsilon_1)$ (which is equal to the realification of $\overline{V(\varepsilon_1)}$) and $\dim_{\mathbb{R}}(E^K) = 2 \cdot \dim_{\mathbb{C}}(V(\varepsilon_1)) = 2$. In this case E carries a G -invariant complex structure J and $E^K = \mathbb{R}v^o + \mathbb{R}Jv^o$. In any case the G -module E is uniquely determined by X .

Let $\tilde{\Phi} : X \rightarrow \tilde{E}$ be another full, isometric, G -equivariant embedding in a Euclidean G -module $\tilde{E}$. Then $\tilde{E}$ and E must be equivalent. We may therefore assume that $E = \tilde{E}$. Since $\tilde{v}^o := \tilde{\Phi}(o) \in E^K$, there exists a G -equivariant isomorphism $f : E \rightarrow E$ of the form $f = a \cdot \text{id}$ (if X is of compact type) or $f = a \cdot \text{id} + b \cdot J$ with $a, b \in \mathbb{R}$, and $f(v^o) = \tilde{v}^o$. Therefore $f \circ \Phi = \tilde{\Phi}$. Since Φ and $\tilde{\Phi}$ are isometries and $\Phi(X)$ and $\tilde{\Phi}(X)$ are both full, f is a linear isometry. $\square$

REMARK 4.3. The proof of Theorem 4.2 shows that in the context of this statement E is irreducible and its complexification $E^{\mathbb{C}}$ is (as a G -module) equivalent to V with $V \cong \overline{V}$ if X is of compact type and to $V \oplus \overline{V}$ with $V \not\cong \overline{V}$ if X splits off a local S^1 -factor, where V is the irreducible complex G -module with highest weight ε_1 .

REMARK 4.4. Let X be a simply connected indecomposable symmetric space with rectangular unit lattice of rank $r \geq 1$, let S be a totally geodesic polysphere (see Theorem 3.8) that contains a maximal torus T_X of X and let Φ the embedding of Theorem 1.1. Then $\Phi(S)$ splits extrinsically as a product of round 2-spheres in 3-dimensional affine Euclidean subspaces.

Indeed, since the maximal torus $\Phi(T_X)$ of $\Phi(S)$ is a Clifford torus, $\Phi(S)$ is of Clifford type. Thus the intrinsic splitting of S into 2-spheres is also extrinsic by Proposition A.10. Now the uniqueness statement of Theorem 1.1 yields the claim.

The uniqueness statement in Theorem 1.1 has the following immediate consequence which is a sharpening of [EQT2] without using classification and case-by-case studies.

COROLLARY 4.5. *Let $X_1 \subset E_1$ and $X_2 \subset E_2$ be two compact extrinsically symmetric spaces (or, equivalently, two submanifolds of Clifford type). Then any isometry $f : X_1 \rightarrow X_2$ extends to a Euclidean map between the affine hulls of X_1 and X_2 . In particular every (intrinsic) isometry of a compact extrinsically symmetric space $X \subset E$ extends to a Euclidean motion of the Euclidean space E .*

5. Eigenvalues of the Laplacian

The knowledge of the Euclidean root datum of a symmetric space X with rectangular unit lattice from Section 3 allows us to compute the eigenvalues of the Laplacian of X by means of FREUDENTHAL's formula. We conclude in particular that the embedding of X constructed in Theorem 1.1 is congruent to an embedding in the first eigenspace in general up to some exceptional cases. A proof of this last result has been outlined earlier by KOBAYASHI-TAKEUCHI [KT] and OHNITA [O] using the classification of standardly

embedded symmetric R -spaces.

Let X be an indecomposable symmetric space with rectangular unit lattice $\Gamma_X \subset \mathfrak{a}$. As in Section 3 let $e_1, \dots, e_r$ be a cubic basis of Γ_X contained in the W_X -orbit of e_1 and let $\varepsilon_1, \dots, \varepsilon_r$ be the corresponding dual basis. By Section 3 we choose a set of positive roots $\mathcal{R}_X^+$ in $\mathcal{R}_X$ such that

$$2\mathcal{R}_X^+ = 2\mathcal{R}_X \cap \{\varepsilon_j, 2\varepsilon_j, \varepsilon_j + \varepsilon_k, \varepsilon_j - \varepsilon_k : 1 \leq j < k \leq r\}.$$

Let $2\rho_X$ be the sum of the positive roots of X counted with their *multiplicities*, that is the (complex) dimension of their root spaces. Let m_1, m_2, m_+ and m_- denote the multiplicities of $\varepsilon_j, 2\varepsilon_j, \varepsilon_j + \varepsilon_k$ and $\varepsilon_j - \varepsilon_k$, respectively (which are independent of j and k), $1 \leq j < k \leq r$. Any of them might be zero and $m_+ = m_- := m_\pm$, unless X is of type $\widehat{A}_{r-1}$ or $\widehat{D}_2$. With $\sum_{1 \leq j < k \leq r} (\varepsilon_j + \varepsilon_k) = (r-1) \sum_{j=1}^r \varepsilon_j$ and $\sum_{1 \leq j < k \leq r} (\varepsilon_j - \varepsilon_k) = \sum_{j=1}^r (r-2j+1)\varepsilon_j$ Proposition 3.4 implies:

LEMMA 5.1.

$$2\rho_X = \begin{cases} \frac{m_-}{2} \sum_{j=1}^r (r-2j+1)\varepsilon_j & \text{in case } \widehat{A}_{r-1}, \\ \sum_{j=1}^r \left(\frac{m_1}{2} + m_2 + m_\pm(r-j) \right) \varepsilon_j & \text{in cases } \widehat{B}_r, \widehat{C}_r, \widehat{D}_r \ (r \geq 3) \text{ and } \widehat{BC}_r, \\ \frac{m_+ + m_-}{2} \varepsilon_1 + \frac{m_+ - m_-}{2} \varepsilon_2, & \text{in case } \widehat{D}_2. \end{cases}$$

Lemma 5.1 enables us to compute the eigenvalues of the Laplacian $\Delta = -\operatorname{div} \circ \operatorname{grad}$ of X . In fact, the complex eigenspaces $E_\lambda(\mathbb{C})$ of Δ decompose into irreducible submodules which, according to Theorem A.3, are (up to equivalence) the V_ω with $\omega \in (\Gamma_X^*)_+$ and each V_ω occurs in precisely one complex eigenspace $E_\lambda(\mathbb{C})$. This λ , which we denote by λ_ω is given by FREUDENTHAL's formula

$$\lambda_\omega = 4\pi^2 \langle \omega + 2\rho_X, \omega \rangle \quad (5)$$

(cf. [T3, Cor. 2, p. 191]). Since Δ is a real operator, each real eigenspace $E_\lambda(\mathbb{R})$ is a real form of $E_\lambda(\mathbb{C})$ and $\{\lambda_\omega : \omega \in (\Gamma_X^*)_+\}$ is the set of eigenvalues of Δ . Combining Lemma 5.1 with FREUDENTHAL's formula (5) and observing that $[0, 1] \rightarrow X, t \mapsto \exp(te_j)p$, are shortest closed geodesic arcs of X of common length $L = \|e_j\| = \frac{1}{\|\varepsilon_j\|}$ ($j = 1, \dots, r$) we get

PROPOSITION 5.2. Let $\omega = \sum_{j=1}^r k_j \varepsilon_j \in (\Gamma_X^*)_+$. Then

$$\lambda_\omega = \begin{cases} \frac{4\pi^2}{L^2} \sum_{j=1}^r k_j \left(k_j + \frac{m_-}{2} (r-2j+1) \right) & \text{in case } \widehat{A}_{r-1}, \\ \frac{4\pi^2}{L^2} \sum_{j=1}^r k_j \left(k_j + \frac{m_1}{2} + m_2 + m_\pm(r-j) \right) & \text{in cases } \widehat{B}_r, \widehat{C}_r, \widehat{D}_r \ (r \geq 3) \text{ and } \widehat{BC}_r, \\ \frac{4\pi^2}{L^2} \left(k_1 \left(k_1 + \frac{m_+ + m_-}{2} \right) + k_2 \left(k_2 + \frac{m_+ - m_-}{2} \right) \right) & \text{in case } \widehat{D}_2. \end{cases}$$

If $\omega = \sum_{j=1}^r k_j \varepsilon_j$ with $k_1, \dots, k_r \in \mathbb{Z}$, then $\omega \in (\Gamma_X^*)_+$ if and only if $k_1 \geq \dots \geq k_r$ in case $\widehat{A}_{r-1}$; $k_1 \geq \dots \geq k_r \geq 0$ in cases $\widehat{B}_r, \widehat{C}_r, \widehat{BC}_r$ and $k_1 \geq \dots \geq k_{r-1} \geq |k_r|$ in case $\widehat{D}_r$. From this we get the following corollary which for irreducible X is equivalent to a result of OHNITA [O, Thm. 7].

COROLLARY 5.3. *The embedding of an indecomposable symmetric space X with rectangular unit lattice given in Theorem 1.1 is congruent to an embedding in the first eigenspace of the Laplacian of X , unless X is of type $\widehat{A}_{r-1}$, $r \geq 2$, with $m_- > 2$ or of type $\widehat{D}_2$ with $|m_+ - m_-| > 2$.*

REMARK 5.4. In the exceptional cases the embedding is not congruent to an embedding in the first eigenspace but rather to an embedding in the k -th eigenspace of the Laplacian, where k might be arbitrary large if only m_- or $|m_+ - m_-|$ are sufficiently large.

PROOF. Since the embedding of Theorem 1.1 maps X into V_{ε_1} , the question remains whether λ_{ε_1} is the first positive eigenvalue of Δ . If X is not of type $\widehat{A}_{r-1}$ or D_2 the answer is apparently ‘yes’. In the following we assume $L = 2\pi$ for simplicity. If X is of type $\widehat{A}_{r-1}$ then

$$\lambda_\omega = \sum_{j=1}^r k_j^2 + \frac{m_-}{2} \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} (k_j - k_{r+1-j})(r+1-2j).$$

Therefore $\lambda_\omega \geq \lambda_{\varepsilon_1}$ if $k_1 - k_r \geq 1$. Thus λ_{ε_1} is minimal if and only if $1 + \frac{m_-}{2}(r-1) \leq \lambda_{\sum_{j=1}^r \varepsilon_j} = r$ which is equivalent to $m_- \leq 2$.

If X is of type $\widehat{D}_2$ we may assume $m_- \leq m_+$ since replacing ε_2 by $-\varepsilon_2$ replaces m_+ by m_- . Then Proposition 5.2 together with $k_1 \geq |k_2|$ implies $\lambda_\omega \geq k_1^2 + k_2^2 + k_1 m_-$ and thus $\lambda_\omega \geq 2 + m_- = \lambda_{\varepsilon_1 - \varepsilon_2}$ if $k_1 \geq 2$ or $k_1 = |k_2| = 1$. Therefore λ_{ε_1} is minimal if and only if $\lambda_{\varepsilon_1} \leq \lambda_{\varepsilon_1 - \varepsilon_2}$ which is equivalent to $m_+ - m_- \leq 2$. $\square$

A. Appendix

A.1. The unit lattice of a compact symmetric space

In this appendix we extend well known relations between unit lattices and roots systems for symmetric spaces of compact type (see e. g. [L1, Prop. 2.4, p. 68 f. and Thm. 3.6, p. 77]) to all compact symmetric spaces.

Let X be a symmetric space that covers a compact symmetric space, $p \in X$, and F_X a maximal flat of X containing p . Let $G = G(X)$, $\pi : G/K \rightarrow X$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{a} \subset \mathfrak{p}$, $\Gamma_X \subset \mathfrak{a}$, $\mathcal{R}_X \subset \mathfrak{a}^*$ and $\mathcal{R}_X^\vee \subset \mathfrak{a}$ as described in Section 2. Let Z be the center of G . Then

$$\{H \in \mathfrak{a} : \exp(H) \in Z\} = \{H \in \mathfrak{a} : \alpha(H) \in \mathbb{Z} \text{ for all } \alpha \in \mathcal{R}_X\}, \quad (6)$$

since for $H \in \mathfrak{a}$, $\exp(H) \in Z$ is equivalent to $\text{Ad}(\exp(H)) = e^{\text{ad}(H)} = \text{id}$ and thus to $e^{\text{ad}(H)}|_{\mathfrak{g}_\alpha} = \text{id}$ for all $\alpha \in \mathcal{R}_X$, that is $e^{2\pi i \alpha(H)} = 1$. Let

$$\Gamma_0 := \Gamma_0(\mathcal{R}_X) := \text{span}_{\mathbb{Z}} \frac{1}{2} \mathcal{R}_X^\vee$$

and

$$\Gamma_1 := \Gamma_1(\mathcal{R}_X) := \frac{1}{2} \{H \in \mathfrak{a} : \alpha(H) \in \mathbb{Z} \text{ for all } \alpha \in \mathcal{R}_X\}.$$

If X is of compact type Γ_0 and Γ_1 are full lattices in $\mathfrak{a}$. However, if X is compact, but not of compact type, then Γ_0 only spans $\mathfrak{a}_0^\perp \subset \mathfrak{a}$, $\mathfrak{a}_0 = \bigcap_{\alpha \in \mathcal{R}_X} \ker(\alpha)$, and Γ_1 is not discrete as $\mathfrak{a}_0 \subset \Gamma_1$. Note that in contrast to Γ_X , Γ_0 and Γ_1 do not change if X is replaced by a symmetric space that covers X or that is covered by X (as $\mathcal{R}_X$ does not change).

THEOREM A.1. *Let X be a symmetric space that covers a compact symmetric space. Then:*

- (i) $2\mathcal{R}_X \subset \Gamma_X^* = \{\alpha \in \mathfrak{a}^* : \alpha(H) \in \mathbb{Z} \text{ for all } H \in \Gamma_X\}$.
- (ii) $\mathcal{R}_X^\vee \subset 2\Gamma_X$.
- (iii) $\Gamma_0 \subset \Gamma_X \subset \Gamma_1$.
- (iv) *The fundamental group $\pi_1(X)$ of X is isomorphic to Γ_X/Γ_0 .*

PROOF. (i) Let $H \in \Gamma_X$. From $\exp(H) \in K \subset G^\sigma$ we get $\exp(-H) = \sigma(\exp(H)) = \exp(H)$ and thus $\exp(2H) = e$. The statement now follows from Equation (6).

- (ii) The unit lattice $\Gamma_{\tilde{X}}$ of the universal cover $\tilde{X}$ of X is contained in Γ_X . Now $\tilde{X}$ splits as $\mathbb{R}^n \times X'$, where X' is a simply connected symmetric space of compact type. Thus $\Gamma_{X'} = \Gamma_{\tilde{X}} \subset \Gamma_X$. Since $\mathcal{R}_{X'} = \mathcal{R}_{\tilde{X}} = \mathcal{R}_X$ and $\Gamma_0(\mathcal{R}_{X'}) = \Gamma_{X'}$ (see [L1, Corollary, p. 77]), the claim follows.
- (iii) This is just a reformulation of (i) and (ii).
- (iv) The fundamental group $\pi_1(X)$ is isomorphic to the group Δ of deck transformations of the universal covering $\tilde{X} \cong \mathbb{R}^n \times X' \rightarrow X$, where X' is a compact simply connected symmetric space. Let $\tilde{p} \in \tilde{X}$ be an element in the fibre over p . As $\Gamma_{\tilde{X}} = \Gamma_{X'}$ and $\Gamma_{X'} = \Gamma_0$ by [L1, Corollary, p. 77]. Thus the maximal flat $F_{\tilde{X}}$ of $\tilde{X}$ corresponding to $\mathfrak{a}$ with $\tilde{p} \in F_{\tilde{X}}$ is isomorphic to $\mathfrak{a}/\Gamma_0$. The universal covering restricts to a covering of the maximal flats $\mathfrak{a}/\Gamma_0 \cong F_{\tilde{X}} \rightarrow T_X \cong \mathfrak{a}/\Gamma_X$ which has deck transformation group Γ_X/Γ_0 . Thus it is sufficient to show that Δ leaves $F_{\tilde{X}}$ invariant. As the elements of $G(\tilde{X})$ push down to isometries of X , $G(\tilde{X})$ normalizes Δ and thus actually centralizes Δ , as Δ is discrete and $G(\tilde{X})$ is connected. Now let $\delta \in \Delta$ and $q \in F_{\tilde{X}}$. Then there exists a maximal flat $F'_{\tilde{X}}$ of $\tilde{X}$ containing q and $\delta(q)$ and $\tau \in G(\tilde{X})$ with $\tau(q) = q$ and $\tau(F'_{\tilde{X}}) = F_{\tilde{X}}$. Thus $\delta(q) = \delta(\tau(q)) = \tau(\delta(q)) \in \tau(F'_{\tilde{X}}) = F_{\tilde{X}}$.

□

A triple $(V, \Gamma, \mathcal{R})$ is called a *Euclidean root datum* if V is a Euclidean vector space, Γ a full lattice in V , and $\mathcal{R}$ a root system in a subspace of V^* , such that $\Gamma_0(\mathcal{R}) \subset \Gamma \subset \Gamma_1(\mathcal{R})$, where $\Gamma_0(\mathcal{R})$ and $\Gamma_1(\mathcal{R})$ are defined above. Two Euclidean root data $(V, \Gamma, \mathcal{R})$ and $(V', \Gamma', \mathcal{R}')$ are called *isomorphic* if there exists a linear isometry $\varphi : V \rightarrow V'$ with $\varphi(\Gamma) = \Gamma'$ and $\varphi(\mathcal{R}) = \mathcal{R}'$, where $\varphi(\alpha) = \alpha \circ \varphi^{-1}$ for $\alpha \in V^*$. For more details see [HQ]. With these definitions Theorem A.1 implies

COROLLARY A.2. *Let $X = G/K$ be a compact symmetric space, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{a} \subset \mathfrak{p}$, $\Gamma_X \subset \mathfrak{a}$ and $\mathcal{R}_X \subset \mathfrak{a}^*$ as above. Then $(\mathfrak{a}, \Gamma_X, \mathcal{R}_X)$ is a Euclidean root datum. Its isomorphism class only depends on X and is called the Euclidean root datum of X . The fundamental group of X is isomorphic to Γ_X/Γ_0 and encoded in the Euclidean root datum.*

A.2. The Theorem of Cartan–Helgason–Takeuchi

Let X be a compact symmetric space and p a base point in X . In this section we denote by G the transvection group of X , that is $G = G(X)$. Let K be the stabilizer of p in G and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding decomposition of the Lie algebra $\mathfrak{g}$ of G . Retaining the notation of Section 2 we choose a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ and denote by $T_X = \exp(\mathfrak{a}).p$ the corresponding maximal torus of X and by $\mathcal{R}_X \subset \mathfrak{a}^*$ the corresponding set of roots of X . We choose a *Weyl chamber* $\mathfrak{a}_+$ of X in $\mathfrak{a}$, that is a connected component of $\mathfrak{a} \setminus (\bigcup_{\alpha \in \mathcal{R}_X} \ker(\alpha))$. Note that W_X acts transitively on the set of all Weyl chambers in $\mathfrak{a}$. Such a choice of $\mathfrak{a}_+$ gives rise to a partial ordering on $\mathfrak{a}^* = \text{Hom}(\mathfrak{a}, \mathbb{R})$ by setting

$$\omega_1 < \omega_2 : \iff (\omega_1(H) < \omega_2(H) \text{ for all } H \in \mathfrak{a}_+), \quad (7)$$

for $\omega_1, \omega_2 \in \mathfrak{a}^*$.

We extend $\mathfrak{a}$ to a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Then $\mathfrak{t} = \mathfrak{a} \oplus (\mathfrak{t} \cap \mathfrak{k})$. One defines the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$, the Weyl chambers of $\mathfrak{g}$ in $\mathfrak{t}$ and the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{t}$ similar to the corresponding notions for X (see e.g. [H3, L1]). We now choose a Weyl chamber $\mathfrak{t}_+$ of $\mathfrak{g}$ in $\mathfrak{t}$ with the property that our previously fixed $\mathfrak{a}_+$ lies in the closure of $\mathfrak{t}_+$. As in Equation (7), $\mathfrak{t}_+$ induces a partial ordering on $\mathfrak{t}^* = \text{Hom}(\mathfrak{t}, \mathbb{R})$. Using the bi-invariant metric on $\mathfrak{g}$ we identify $\mathfrak{a}$ with $\mathfrak{a}^*$ and $\mathfrak{t}$ with $\mathfrak{t}^*$. Note that this identification of $\mathfrak{a}^*$ with $\mathfrak{a}$ is W_X -equivariant and that $\mathfrak{a}^*$ becomes the subset $\{\mu \in \mathfrak{t}^* : \mu(\mathfrak{t} \cap \mathfrak{k}) = \{0\}\}$ of $\mathfrak{t}^*$, and the partial orderings on $\mathfrak{a}^*$ induced by $\mathfrak{a}_+$ and by $\mathfrak{t}_+$ coincide. We denote by $\mathfrak{a}_+^*$ the image of $\mathfrak{a}_+$ in $\mathfrak{t}^*$ under this identification and by $\text{cl}(\mathfrak{a}_+^*)$ its closure. We set $\Gamma_X^* = \{\omega \in \mathfrak{a}^* : \omega(\Gamma_X) \subset \mathbb{Z}\}$ and $(\Gamma_X^*)_+ = \Gamma_X^* \cap \text{cl}(\mathfrak{a}_+^*)$. If $\Gamma_G = \{H \in \mathfrak{t} : \exp(H) = e\}$ and $\Gamma_G^* = \{\omega \in \mathfrak{t}^* : \omega(\Gamma_G) \subset \mathbb{Z}\}$ and $(\Gamma_G^*)_+ = \Gamma_G^* \cap \text{cl}(\mathfrak{t}_+^*)$ then (cf. [T3, Lem. 1, p. 143])

$$(\Gamma_X^*)_+ \subset (\Gamma_G^*)_+.$$

Let V be a complex G -module. We endow V with a G -invariant hermitian inner product $(\cdot, \cdot)_{\mathbb{C}}$. For an element $\omega \in \mathfrak{t}^*$ we set

$$V_{\omega} := \left\{ v \in V : \exp(H).v = e^{2\pi i \omega(H)} v \text{ for all } H \in \mathfrak{t} \right\}.$$

The elements of $\Omega_V := \{\omega \in \mathfrak{t}^* : V_\omega \neq \{0\}\}$ are called *weights* of V and the decomposition

$$V = \bigoplus_{\omega \in \Omega_V} V_\omega \quad (8)$$

is called the *weight space decomposition* of V . It is orthogonal with respect to $(\cdot, \cdot)_{\mathbb{C}}$. We have $\Omega_V \subset \Gamma_G^*$. Note that the roots of a compact Lie algebra $\mathfrak{g}$ are the weights of the adjoint representation of the group of inner automorphisms of $\mathfrak{g}$.

A *K-spherical G-module* is an irreducible complex G -module V such that $V^K := \{v \in V : k.v = v \text{ for all } k \in K\}$ has positive dimension. By a result of CARTAN V^K has dimension one [Ca, I.5 and V.17] (see also [T3, Thm. 5.5, p. 92]). We denote by $\widehat{G}_K$ set of all equivalence classes of K -spherical G -modules.

THEOREM A.3 (Cartan–Helgason–Takeuchi, [T3, Thm. 8.2]). *The map*

$$(\Gamma_X^*)_+ \rightarrow \widehat{G}_K, \quad \omega \mapsto [V(\omega)]$$

is a bijection. Here $V(\omega)$ denotes an irreducible complex G -module with highest weight ω and $[V(\omega)]$ its equivalence class.

NOTE A.4. Theorem A.3 has a long history (c. f. BOREL [B, Chap. IV, B, §4]). After previous work by CARTAN [Ca], HELGASON [H2] (see also [H4, Ch. V, §4, 1.]) gave a proof for semi-simple symmetric spaces. The extension to all compact symmetric spaces is due to TAKEUCHI [T3].

A.3. Submanifolds of Clifford Type and their splittings

Let $r > 0$ be an integer and $\rho_1, \dots, \rho_r$ be positive real numbers. A space

$$C' := \{z = (z_1, \dots, z_r) \in \mathbb{C}^r : |z_j| = \rho_j \text{ for all } j = 1, \dots, r\}$$

is called a *standard Clifford torus*. Endowed with the inner product $\langle z, w \rangle = \operatorname{Re} \left(\sum_{j=1}^r z_j \overline{w_j} \right)$ $\mathbb{C}^r$ is a Euclidean space.

A submanifold C of a Euclidean space E is called a *Clifford torus* if it is the image of a standard Clifford torus $C' \subset \mathbb{C}^r$ under an affine isometric map $\varphi : \mathbb{C}^r \rightarrow E$. Let $p = \varphi(w) \in C$. Then the unit lattice $\Gamma = \{v \in T_p C : \operatorname{Exp}_p(v) = p\}$, where Exp_p denotes the Riemannian exponential map of C at p , has an orthogonal basis $b_1, \dots, b_r$, which is unique up to sign and order (see [EHQ, Lemma 1.1]). Indeed, Γ is the direct sum of the unit lattices of the circles $C_k(p) = \varphi(\{z \in \mathbb{C}^r : |z_k| = \rho_k \text{ and } z_j = w_j \text{ for all } k \neq j\})$, $k = 1, \dots, r$, which lie in the pairwise orthogonal affine planes $E_k(p) = \varphi\{z \in \mathbb{C}^r : z_j = w_j \text{ for all } k \neq j\}$.

EXAMPLE A.5. Let V be a Euclidean space. We consider $V^c = V \otimes \mathbb{C}$ as a *real* Euclidean space by taking the real part of the hermitian extension of the inner product on V . Let $v, w \in V^c$ be orthogonal with respect to the hermitian inner product and $v \neq 0$. Then $C = \{zv + \bar{z}w : z \in \mathbb{C} \text{ with } |z| = 1\}$ is a planar circle in V^c . In fact, for $z = c + is$ with $c = \cos t$, $s = \sin t$ we have

$$zv + \bar{z}w = (c + is)v + (c - is)w = c(v + w) + s i(v - w)$$

which is a circle since $v + w$ and $i(v - w)$ have the same length and are perpendicular with respect to the (real) Euclidean inner product on V^c .

More generally, if $v_1, \dots, v_r, w_1, \dots, w_r \in V^c$ are pairwise orthogonal with respect to the hermitian inner product and $v_1, \dots, v_r$ are all nonzero, then

$$C := \left\{ \sum_{j=1}^r (z_j v_j + \bar{z}_j w_j) : z_j \in \mathbb{C} \text{ with } |z_j| = 1 \right\}$$

is a Clifford torus in V^c .

As in [EHQ] we call an embedded submanifold $X \subset E$ of a Euclidean space E of *Clifford type*, if every geodesic of X lies in a totally geodesic submanifold C of X which is a Clifford torus in E .

THEOREM A.6 ([EHQ, Corollary 2.8]). *Let $X \subset E$ be a connected submanifold of a Euclidean space E . Then X is a compact extrinsically symmetric space if and only if X is of Clifford type. In this case, X is intrinsically a compact symmetric space with rectangular unit lattice and all maximal tori of X are Clifford tori in E .*

Intrinsic and extrinsic splittings

Let X be a connected Riemannian manifold. Recall that X *splits intrinsically* as $X_1 \times \dots \times X_k$ if $X_1, \dots, X_k$ are Riemannian manifolds and there exists an isometry $f : X_1 \times \dots \times X_k \rightarrow X$. Such an isometry f is called an (explicit) splitting and gives rise to k pairwise orthogonal foliations whose leaves through $p = f(x_1^0, \dots, x_k^0)$ are $X_j(p) := f(x_1^0, \dots, x_{j-1}^0, X_j, x_{j+1}^0, \dots, x_k^0)$ for $1 \leq j \leq k$.

Let Y be another connected Riemannian manifold and $\iota : X \rightarrow Y$ an isometric embedding. We say that the submanifold $\iota(X) \subset Y$ *splits extrinsically* as $X_1 \times \dots \times X_k$ if there exists a commutative diagram

$$\begin{array}{ccc} Y_1 \times \dots \times Y_k & \xrightarrow{F} & Y \\ \uparrow \iota_1 \times \dots \times \iota_k & & \uparrow \iota \\ X_1 \times \dots \times X_k & \xrightarrow{f} & X \end{array}$$

where the first and the last rows are (explicit) intrinsic splittings and the maps $\iota_j : X_j \rightarrow Y_j$, $j = 1, \dots, k$, are isometric embeddings. In this case the foliations corresponding to intrinsic splittings of X and Y satisfy $\iota(X_j(p)) \subset Y_j(\iota(p))$ for all $p \in X$ and for all $1 \leq j \leq k$.

LEMMA A.7. *Let $\iota : X \rightarrow Y$ be an isometric embedding and let $f : X_1 \times \dots \times X_k \rightarrow X$ and $F : Y_1 \times \dots \times Y_k \rightarrow Y$ be explicit intrinsic splitting that satisfy $\iota(X_j(p)) \subset Y_j(\iota(p))$ for all $p \in X$ and for all $1 \leq j \leq k$. Then $\iota(X)$ splits extrinsically as $X_1 \times \dots \times X_k$. More precisely, there exist isometric embeddings $\iota_j : X_j \rightarrow Y_j$, $j = 1, \dots, k$, such that $F \circ (\iota_1 \times \dots \times \iota_k) = \iota \circ f$.*

PROOF. For simplicity we may assume $X = X_1 \times \cdots \times X_k$, $Y_1 = Y_1 \times \cdots \times Y_k$, $f = \text{id}_X$ and $F = \text{id}_Y$. Let $\iota'_j = \text{pr}_j \circ \iota$, where $\text{pr}_j : Y \rightarrow Y_j$ is the j -th projection. Since for instance $X_1(p) \subset Y_1(\iota(p))$ for all $p = (x_1, \dots, x_k) \in X$, we get

$$\iota(X_1, x_2, \dots, x_k) \subset (Y_1, \iota'_2(x_1, \dots, x_k), \dots, \iota'_k(x_1, \dots, x_k)).$$

Since the left hand side is independent of x_1 and not contained in any other leaf $Y_1(q)$, the maps $\iota'_2, \dots, \iota'_k$ are independent of x_1 . Similarly, ι'_j is independent of x_ℓ for all $\ell \neq j$. Thus every ι'_j induces a mapping $\iota_j : X_j \rightarrow Y_j$ and $\iota = \iota_1 \times \cdots \times \iota_k$. $\square$

We immediately get

COROLLARY A.8. *Let $X \subset E$ be an embedded submanifold of a Euclidean space E and $f : X_1 \times X_k \rightarrow X$ an intrinsic splitting of X . If*

(i) *$\text{aff}(X_j(p))$ is perpendicular to $\text{aff}(X_\ell(p))$ and*

(ii) *$\text{aff}(X_j(p))$ is parallel to $\text{aff}(X_j(q))$*

for all $p, q \in X$ and $1 \leq j < \ell < k$ then X splits extrinsically as $X_1 \times \cdots \times X_k$ and vice-versa. Here $\text{aff}(M)$ denotes the affine hull of a subset $M \subset E$.

For Clifford tori we get:

LEMMA A.9. *If a Clifford torus $C \subset E$ splits intrinsically as $X_1 \times \cdots \times X_k$, then this splitting is also extrinsic and $X_j(p)$ are Clifford tori for all $j = 1, \dots, k$ and all $p \in C$.*

PROOF. It suffices to assume $k = 2$ and $C = X_1 \times X_2$. Let $p \in C$. Then the unit lattice $\Gamma \subset T_p C$ of C at p has an orthogonal basis $b_1, \dots, b_r$ with $|b_1| \leq |b_2| \leq \cdots \leq |b_r|$. Since $X_1(p)$ and $X_2(p)$ are compact totally geodesic submanifolds of C , they are both tori and we get an orthogonal decomposition of the unit lattice Γ of C at p into $\Gamma^1 \oplus \Gamma^2$, where Γ^1 is the unit lattice of $X_1(p)$ and Γ^2 the unit lattice of $X_2(p)$. Let $j \in \{1, \dots, r\}$. From $b_j = b_j^1 + b_j^2$ with $b_j^1 \in \Gamma^1$ and $b_j^2 \in \Gamma^2$ we get $|b_j|^2 = |b_j^1|^2 + |b_j^2|^2$. Thus $b_1 \in \Gamma^1$ or $b_1 \in \Gamma^2$, since b_1 is a nonzero element of shortest length in Γ (see also part (ii) in the proof of Proposition 3.1). By induction we get $b_j \in \Gamma^1 \cup \Gamma^2$ for all $j \in \{1, \dots, r\}$. Hence Γ^1 and Γ^2 are rectangular lattices and the $X_1(p)$ resp. $X_2(p)$ are Clifford tori in orthogonal subspaces $E_1(p)$ resp. $E_2(p)$. Since $E_j(p)$ is parallel to $E_j(q)$ for all $j = 1, 2$ and $p, q \in C$, C splits extrinsically by Corollary A.8. $\square$

Lemma A.9 generalizes to submanifolds of Clifford type:

PROPOSITION A.10. *Let $X \subset E$ be a submanifold of Clifford type. If X splits intrinsically as a Riemannian product $X_1 \times \cdots \times X_k$, then this splitting is extrinsic.*

PROOF. Since X is a compact symmetric space, the same holds for all factors X_j . It suffices to assume $k = 2$ and $X = X_1 \times X_2$. Then for a point $p = (p_1, p_2) \in X$ we have $X_1(p) = X_1 \times \{p_2\}$ and $X_2(p) = \{p_1\} \times X_2$. We set $E_1(p) = \text{aff}(\{p_1\}X_1(p))$ and $E_2(p) = \text{aff}(X_2(p))$. Now for any two points in an X_j there exists a maximal torus of X_j containing these two points. Thus we have

$$E_1(p) = \text{aff} \bigcup_{T_1 \ni p_1} (T_1 \times \{p_2\}),$$

where the union is taken over all maximal tori of X_1 that contain p_1 and

$$E_2(p) = \text{aff} \bigcup_{T_2 \ni p_2} (\{p_1\} \times T_2),$$

where the union is taken over all maximal tori of X_2 that contain p_2 .

If T_j , $j = 1, 2$, are maximal tori of X_j with $p_j, p'_j \in T_j$ then $T = T_1 \times T_2$ is a maximal torus of X and hence a Clifford torus in E . By Lemma A.9 and Corollary A.8 the affine hulls of $T_1 \times \{p_2\}$ and $\{p_1\} \times T_2$ are perpendicular while those of $T_1 \times \{p_2\}$ and $T_1 \times \{q_2\}$ (resp. $\{p_1\} \times T_2$ and $\{q_1\} \times T_2$) are parallel. Thus $E_1(p)$ is perpendicular to $E_2(p)$ for all p and $E_1(p)$ is parallel to $E_1(q)$ (resp. $E_2(p)$ is parallel to $E_2(q)$) for all $p, q \in X$. The assertion now follows from Corollary A.8. $\square$

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