COTILTING INVARIANCE OF THE AUSLANDER-REITEN CONJECTURE

KAMRAN DIVAANI-AAZAR$^{(*)}$, ALI MAHIN FALLAH AND MASSOUD TOUSI

Abstract. Let R be an associative ring with identity, and let T be a tilting right R-module, with $S = \mathrm{End}(T)$. It is known that if R is a Noetherian algebra that satisfies the Auslander-Reiten conjecture, then so is S. In this paper, we consider the dual situation where C is a cotilting right R-module, with $S = \mathrm{End}(C)$. We investigate the invariance of the property of satisfying the Auslander-Reiten conjecture when passing from R to S.

1. Introduction

Nakayama's conjecture [25], a long-standing open problem in the representation theory of Artin algebras, asserts that an Artin algebra R is self-injective if all terms in its minimal injective resolution are projective. In their study of this conjecture, Auslander and Reiten [6] conjectured that for an Artin algebra R, every indecomposable injective left R-module appears as a direct summand in one of the terms of the minimal injective resolution of R. This conjecture is now known as the generalized Nakayama conjecture. Auslander and Reiten [6] further proved that the generalized Nakayama conjecture holds for all Artin algebras if and only if the following conjecture is true for all Artin algebras. Throughout, the term *ring* will mean an associative ring with identity.

Conjecture (ARC): Let R be a ring, and M a finitely generated left R-module. If $\mathrm{Ext}^i_R(M, M \oplus R) = 0$ for all $i > 0$, then M is projective.

This conjecture, now known as the Auslander–Reiten conjecture, is a consequence of the finitistic dimension conjecture. The finitistic dimension conjecture states that, for an Artin algebra R, the supremum of the projective dimensions of all finitely generated left R-modules with finite projective dimension is finite. If the finitistic dimension conjecture holds for all Artin algebras, then (ARC) also holds for all Artin algebras.

2020 *Mathematics Subject Classification.* 16Gxx; 16D20; 16Exx; 16E30.

Key words and phrases. Auslander-Reiten conjecture; co-Noetherian ring; cotilting module; Noetherian algebra; product-complete module; self-orthogonal module; tilting module; Wakamatsu tilting module.

The research of the second author is supported by a grant from IPM (No.1402160015).

$^{(*)}$ Corresponding author.

Tilting theory is crucial in the representation theory of Artin algebras. Miyashita [24] explored the dual concept of cotilting modules over Noetherian rings in 1986. Auslander and Reiten further investigated cotilting modules over Artin algebras, viewing them as a non-commutative analogue of dualizing modules over commutative Noetherian rings [7, 8]. For Artin algebras, these definitions coincide.

Wei proved in [29] that (ARC) is preserved under tilting equivalences of Artin algebras. Specifically, if T is a tilting right module over an Artin algebra R with the endomorphism ring $S = \mathrm{End}(T)$, and if R satisfies (ARC), then so does S. The second author generalized this result to Noetherian algebras in [18]. Furthermore, Wei showed in [29] that (ARC) is also preserved under cotilting equivalences of Artin algebras. That is, if C is a cotilting right module over an Artin algebra R with the endomorphism ring $S = \mathrm{End}(C)$, and if R satisfies (ARC), then S does as well. This raises the natural question of whether a similar preservation property holds for Noetherian algebras. To address this question, we recall the following conjecture from [11], which serves as a dual to (ARC):

Conjecture (DARC): Let R be a ring, and M an Artinian left R-module. If $\mathrm{Ext}^i_R(M \oplus I, M) = 0$ for every injective left R-module I and all $i > 0$, then M is injective.

This conjecture is motivated by the equivalence between (DARC) and (ARC) for Artin algebras. An Artin algebra R satisfies (ARC) if and only if R^{op} satisfies (DARC). Furthermore, if a Noetherian algebra R over a commutative Noetherian complete semi-local ring satisfies (ARC), then R^{op} satisfies (DARC). As our main result, we prove:

Theorem 1.1. *Let R be a right Artinian ring, let C be a cotilting right R-module, and set $S = \mathrm{End}(C_R)$.*

1. *If S satisfies (ARC), then R^{op} satisfies (DARC).*
2. *The converse holds under the additional assumptions that R is right co-Noetherian and S is left Artinian.*

We prove the above result in Theorem 3.13. Clearly, the above result generalizes Wei's aforementioned result; see [29, Main Theorem and Corollary 3.7].

We end the paper by presenting an example showing that there exist rings S satisfying (ARC) by Theorem 1.1 whenever the ring R^{op} satisfies (DARC).

2. Preliminaries

Throughout, R is an associative ring with identity. We use the notation M_R (respectively, ${}_RM$) to denote a right (respectively, left) R-module. The category of all (respectively, finitely generated) right R-modules is denoted by Mod-R (respectively, mod-R), while R-Mod (respectively, R-mod) represents the category of all (respectively, finitely generated) left R-modules.

Let M_R be an R-module. We denote by $\mathrm{Add}(M)$ (respectively, $\mathrm{add}(M)$) the class of right R-modules that are isomorphic to a direct summand of a direct sum of copies (respectively, finitely many copies) of M. Dually, $\mathrm{Prod}(M)$ denotes the class of right R-modules that are isomorphic to a direct summand of a direct product of copies of M.

Let $\mathscr{C}$ be a full subcategory of the category of Mod-R, and n a non-negative integer. We denote by $\widehat{\mathscr{C}}_n$ the category of all R-modules M_R such that there exists an exact sequence

$$0 \to C_n \to \cdots \to C_1 \to C_0 \to M \to 0$$

with each $C_i \in \mathscr{C}$. Dually, we then denote by $\check{\mathscr{C}}_n$ the category of all R-modules M_R such that there exists an exact sequence

$$0 \to M \to C^0 \to C^1 \to \cdots \to C^n \to 0$$

with each $C^i \in \mathscr{C}$.

For an R-module T_R, denote by $\mathrm{gen}^*(T_R)$ the class consisting of the R-modules N_R for which there exists an exact sequence of the form

$$\cdots \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} N \longrightarrow 0$$

with each $T_i \in \mathrm{add}(T)$ and $\mathrm{Ext}^1_R(T, \mathrm{Ker} f_i) = 0$ for all $i \geqslant 0$. Dually, $\mathrm{cogen}^*(T_R)$ denotes the class of all R-modules N_R for which there exists an exact sequence of the form

$$0 \longrightarrow N \xrightarrow{f^{-1}} T^0 \xrightarrow{f^0} T^1 \xrightarrow{f^1} \cdots$$

with each $T^i \in \mathrm{add}(T)$ and $\mathrm{Ext}^1_R(\mathrm{Coker} f^i, T) = 0$ for all $i \geqslant -1$.

For an R-module M_R, the symbol $M^{\perp}$ denotes the subcategory of R-modules N_R such that $\mathrm{Ext}^i_R(M, N) = 0$ for all $i \geqslant 1$. Dually, ${}^{\perp}M$ denotes the subcategory of R-modules N_R such that $\mathrm{Ext}^i_R(N, M) = 0$ for all $i \geqslant 1$.

For an R-module M_R and a positive integer n, the symbol $\Omega^n_R M$ denotes the n-th syzygy of M, with the convention that $\Omega^0_R M = M$. We also recall the definition of cosyzygies. Given an R-module M_R, and an injective resolution

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^{i-1} \to E^i \to \cdots$$

for M, the n-th cosyzygy of M is defined as $\mho_n(M) = \mathrm{Im}(E^{n-1} \to E^n)$ for every $n \geqslant 1$. Additionally, we set $\mho_0(M) = M$.

The following lemma will be useful in our investigation. Its proof is an immediate consequence of the dimension-shift property of Ext modules and is therefore omitted.

Lemma 2.1. *Let M_R, N_R and X_R be R-modules, and $n \in \mathbb{N}$.*

(i) *If $0 \to M \to K_0 \to K_1 \to \cdots \to K_{n-1} \to X \to 0$ is an exact sequence with each K_j in $N^{\perp}$, then $\operatorname{Ext}^{i+n}_R(N, M) \cong \operatorname{Ext}^i_R(N, X)$ for all $i \geqslant 1$.*

(ii) *If $0 \to X \to L_0 \to L_1 \to \cdots \to L_{n-1} \to M \to 0$ is an exact sequence with each L_j in ${}^{\perp}N$, then $\operatorname{Ext}^{i+n}_R(M, N) \cong \operatorname{Ext}^i_R(X, N)$ for all $i \geqslant 1$.*

Recall that an R-module M_R is called *self-orthogonal* if $\operatorname{Ext}^i_R(M, M) = 0$ for all $i \geqslant 1$. We next recall the definitions of tilting and Wakamatsu tilting modules.

Definition 2.2. Let $n \in \mathbb{N}_0$. A finitely generated self-orthogonal R-module T_R is called *n-tilting* if

(i) $\operatorname{pd}(T_R) \leqslant n$, and

(ii) $R \in \widetilde{\operatorname{add}(T)}_n$.

Definition 2.3. A self-orthogonal R-module T_R is called *Wakamatsu tilting* if

(i) $T_R \in \operatorname{gen}^*(R)$, and

(ii) $R \in \operatorname{cogen}^*(T_R)$.

The definitions of tilting left modules and Wakamatsu tilting left modules are similar. It is straightforward to verify that every tilting module is also a Wakamatsu tilting module. According to [28, Corollary 3.2], Wakamatsu tilting modules can be characterized as follows:

Lemma 2.4. *For a bimodule ${}_ST_R$, the following are equivalent:*

(i) *T_R is a Wakamatsu tilting module with $S \cong \operatorname{End}(T_R)^{\mathrm{op}}$;*

(ii) *${}_ST$ is a Wakamatsu tilting module with $R \cong \operatorname{End}({}_ST)^{\mathrm{op}}$;*

(iii) *One has*

  (1) *$T_R \in \operatorname{gen}^*(R)$ and ${}_ST \in \operatorname{gen}^*(S)$.*

  (2) *$S \cong \operatorname{End}(T_R)^{\mathrm{op}}$ and $R \cong \operatorname{End}({}_ST)^{\mathrm{op}}$.*

  (3) *The modules T_R and ${}_ST$ are self-orthogonal.*

There are several definitions of cotilting modules. In this paper, we adopt the definition provided by Miyashita in 1986; see [24, page 142]. This notion of cotilting modules is also discussed in [15, 22, 23].

Definition 2.5. Let C_R be a finitely generated R-module and $S = \operatorname{End}(C_R)$. Assume that the rings R and S are right Noetherian and left Noetherian; respectively. We say C_R is *cotilting* if it is a Wakamatsu tilting R-module and both $\operatorname{id}(C_R)$ and $\operatorname{id}({}_SC)$ are finite.

Left cotilting modules are defined analogously. Below, we present some typical examples of cotilting modules.

Example 2.6. (i) For an Artin algebra R and a tilting left R-module T, the dual of T is a cotilting right R-module.

(ii) Let R be an Iwanaga–Gorenstein ring. Then R, regarded as either a right or a left R-module, is cotilting.

(iii) Let Γ be a Cohen-Macaulay local ring with a dualizing module ω, and R be a Γ-order (i.e. R is a Γ-algebra and is maximal Cohen-Macaulay as a Γ-module). Then, by [23, Proposition 2.12], $\mathrm{Hom}_\Gamma(R,\omega)$ is a cotilting right R-module.

(iv) Let C_R be a cotilting R-module, and let $S = \mathrm{End}(C_R)$. Then, the rings $\widetilde{R} = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$ and $\widetilde{S} = \begin{pmatrix} S & S \\ 0 & S \end{pmatrix}$ are right Noetherian and left Noetherian; respectively. Set $\widetilde{C} = (0 \to C) \oplus (C \to C)$. By applying the arguments from Lemmas 3.4 and 3.7 in [31] and using Lemma 2.4, we can conclude that $\widetilde{C}_{\widetilde{R}}$ is a cotilting $\widetilde{R}$-module and $\widetilde{S} \cong \mathrm{End}(\widetilde{C}_{\widetilde{R}})^{\mathrm{op}}$.

3. Main Result

To prove Theorem 1.1, we require eight lemmas. We begin with the following result, which is essential for its proof.

Lemma 3.1. *Let C_R be a cotilting R-module with $S = \mathrm{End}(C_R)$. Then there exists a duality of categories:*

$$^{\perp}(C_R) \cap \mathrm{mod}\text{–}R \underset{\mathrm{Hom}_S(-,C)}{\overset{\mathrm{Hom}_R(-,C)}{\rightleftarrows}} {}^{\perp}({}_SC) \cap S\text{–}\mathrm{mod}.$$

Furthermore, for any $X, Y \in {}^{\perp}(C_R) \cap \mathrm{mod}\text{–}R$ and every $i \geqslant 0$, we have

$$\mathrm{Ext}^i_R(X,Y) \cong \mathrm{Ext}^i_S(\mathrm{Hom}_R(Y,C), \mathrm{Hom}_R(X,C)).$$

Proof. It follows by [28, Theorem 4.2 and Propositions 5.6 and 4.5]. □

Definition 3.2. An R-module C_R is called *product-complete* if $\mathrm{Prod}(C) \subseteq \mathrm{Add}(C)$.

As mentioned in [19, Remark 3.5], an R-module C_R is product-complete if and only if $\mathrm{Prod}(C) = \mathrm{Add}(C)$. For an R-module C_R where the ring $S = \mathrm{End}(C_R)$ is left Noetherian, by [17, Corollary 4.4], C is product-complete if and only if ${}_SC$ has finite length.

Lemma 3.3. *Let C_R be a self-orthogonal R-module and $n \in \mathbb{N}_0$.*

(i) $\widetilde{\mathrm{add}(C)}_n \cap C^{\perp} = \mathrm{add}(C) = \widehat{\mathrm{add}(C)}_n \cap {}^{\perp}C$.

(ii) *Moreover, if C_R is product-complete, then*

$$\widetilde{\mathrm{Add}(C)}_n \cap C^{\perp} = \mathrm{Add}(C) = \widehat{\mathrm{Add}(C)}_n \cap {}^{\perp}C.$$

Proof. (i) As the proofs of these equalities are similar, we only prove the equality on the left side. Clearly,

$$\operatorname{add}(C) \subseteq \widetilde{\operatorname{add}(C)}_n \cap C^{\perp}.$$

Assume that $N \in \widetilde{\operatorname{add}(C)}_n \cap C^{\perp}$. We will show that $N \in \operatorname{add}(C)$ by induction on n. The claim is obvious for $n = 0$. If $n = 1$, then we have an exact sequence

$$0 \to N \to C^0 \to C^1 \to 0,$$

where C^0 and C^1 are in $\operatorname{add}(C)$. Since $N \in C^{\perp}$ and $C^1 \in \operatorname{add}(C)$, this short exact sequence splits, and so $N \in \operatorname{add}(C)$.

Now, assume that $n > 1$ and the conclusion holds for $n-1$. We have an exact sequence

$$0 \to N \xrightarrow{f^{-1}} C^0 \xrightarrow{f^0} C^1 \to \cdots \xrightarrow{f^{n-1}} C^n \to 0,$$

with $C^i \in \operatorname{add}(C)$ for all $i = 0, 1, \ldots, n$. Put $L = \operatorname{Im}(f^0)$. Since $N \in C^{\perp}$, it follows that $L \in C^{\perp}$. So, $L \in \widetilde{\operatorname{add}(C)}_{n-1} \cap C^{\perp}$. Hence, by the induction hypothesis, $L \in \operatorname{add}(C)$. Consequently, $N \in \operatorname{add}(C)$ by the case $n = 1$.

(ii) Since C_R is product-complete, we have $\operatorname{Prod}(C) = \operatorname{Add}(C)$. Using this fact together with the self-orthogonality of C_R, it follows readily that $\operatorname{Ext}^i_R(X, Y) = 0$ for all X and Y in $\operatorname{Add}(C)$ and every $i \geqslant 1$. We now proceed exactly as in part (i). □

Lemma 3.4. *Let C_R be a cotilting R-module, M_R a finitely generated R-module and $n \in \mathbb{N}_0$. If $M \in {}^{\perp}C$ and $\operatorname{id}(M_R) \leqslant n$, then $M \in \widetilde{\operatorname{add}(C)}_n$.*

Proof. By [28, Propositions 5.6], there exists an exact sequence

$$0 \longrightarrow M \xrightarrow{f^{-1}} C^0 \xrightarrow{f^0} C^1 \xrightarrow{f^1} C^2 \longrightarrow \cdots$$

where $C^i \in \operatorname{add}(C)$ and $\operatorname{Im}(f^i) \in {}^{\perp}C$ for all $i \geqslant 0$. For simplicity, we put $K_0 = M$ and $K_{i+1} = \operatorname{Im}(f^i)$ for each $i \geqslant 0$. It is fairly easy to verify that $C^i \in (K_{n+1})^{\perp}$ for every $i = 0, \ldots, n-1$. So, by Lemma 2.1(i), we have the following isomorphism

$$\operatorname{Ext}^1_R(K_{n+1}, K_n) \cong \operatorname{Ext}^{n+1}_R(K_{n+1}, M) = 0.$$

Hence, the short exact sequence

$$0 \to K_n \to C^n \to K_{n+1} \to 0$$

splits, and so $K_n \in \operatorname{add}(C)$. Consequently, $M \in \widetilde{\operatorname{add}(C)}_n$. □

Lemma 3.5. *Let C_R be a cotilting R-module with $\operatorname{id}(C_R) = n$. Then every injective right R-module belongs to $\widehat{\operatorname{Add}(C)}_n$.*

Proof. Let $S = \mathrm{End}(C_R)$, and let I_R be an injective R-module. By [13, Theorem 2.7], we have $n = \mathrm{id}({}_SC)$. In view of [10, Theorem 3.2.13], it is straightforward to see that

$$\mathrm{fd}(\mathrm{Hom}_R(C,I)_S) \leqslant \mathrm{id}({}_SC) = n.$$

Thus, we have an exact sequence

$$0 \to F_{n+1} \to S^{(Jn)} \to S^{(Jn-1)} \to S^{(J_{n-2})} \to \cdots \to S^{(J_0)} \to \mathrm{Hom}_R(C,I) \to 0, \tag{3.1}$$

where F_{n+1} is a flat right S-module. As the R-module I_R is injective, by the Hom-evaluation isomorphism, one has

$$\mathrm{Tor}^S_{i\geqslant 1}(\mathrm{Hom}_R(C,I),C) \cong \mathrm{Hom}_R(\mathrm{Ext}_S^{i\geqslant 1}(C,C),I) = 0$$

and

$$\mathrm{Hom}_R(C,I) \otimes_S C \cong I.$$

So, applying the functor $-\otimes_S C$ to (3.1) yields the following exact sequence

$$0 \longrightarrow F_{n+1} \otimes_S C \longrightarrow C^{(J_n)} \xrightarrow{d_n} C^{(J_{n-1})} \xrightarrow{d_{n-1}} C^{(J_{n-2})} \longrightarrow \cdots \longrightarrow C^{(J_0)} \xrightarrow{d_0} I \longrightarrow 0.$$

Set $K_0 = I$ and $K_{j+1} = \mathrm{Ker}\, d_j$ for every $j = 0, ..., n$. By [10, Theorem 3.2.15], we conclude that

$$\mathrm{Ext}^i_R(C^{(J)}, K_{n+1}) \cong \mathrm{Ext}^i_R(C, K_{n+1})^J \cong (F_{n+1} \otimes_S \mathrm{Ext}^i_R(C,C))^J = 0, \tag{3.2}$$

for every set J and all $i \geqslant 1$.

By [10, Theorem 3.2.15], we have

$$\mathrm{id}((K_{n+1})_R) = \mathrm{id}((F_{n+1} \otimes_S C)_R) \leqslant \mathrm{id}(C_R) = n. \tag{3.3}$$

Hence, by applying Lemma 2.1(ii) to the exact sequence

$$0 \longrightarrow K_n \longrightarrow C^{(J_{n-1})} \xrightarrow{d_{n-1}} C^{(J_{n-2})} \longrightarrow \cdots \longrightarrow C^{(J_0)} \xrightarrow{d_0} I \longrightarrow 0$$

and using (3.3), we obtain that

$$\mathrm{Ext}^1_R(K_n, K_{n+1}) \cong \mathrm{Ext}^{n+1}_R(I, K_{n+1}) = 0.$$

Thus, the short exact sequence

$$0 \to K_{n+1} \to C^{(J_n)} \to K_n \to 0$$

splits, and so $K_n \in \mathrm{Add}(C)$. □

Lemma 3.6. *Let C_R be a finitely presented self-orthogonal R-module. Then, for any non-negative integer n, the classe $\widehat{\mathrm{Add}(C)}_n$ is closed under extensions.*

Proof. Assume that $0 \to X \to Y \to Z \to 0$ is an exact sequence with X and Z in $\widehat{\mathrm{Add}(C)}_n$. We claim by induction on n that Y in $\widehat{\mathrm{Add}(C)}_n$. Let $n = 0$. Since C_R is a finitely presented self-orthogonal module and also X and Z are in $\widehat{\mathrm{Add}(C)}_0 = \mathrm{Add}(C)$, it follows that $\mathrm{Ext}^1_R(Z, X) = 0$. Hence, $Y \cong X \oplus Z$, and so $Y \in \widehat{\mathrm{Add}(C)}_0$.

Next, let $n > 0$ and assume that the claim holds for $n-1$. Since X and Z are in $\widehat{\mathrm{Add}(C)}_n$, we have exact sequences

$$0 \to K \to C_0 \to X \to 0$$

and

$$0 \to L \to C'_0 \to Z \to 0$$

with C_0 and C'_0 in $\mathrm{Add}(C)$ and K and L in $\widehat{\mathrm{Add}(C)}_{n-1}$. Consider the following pull-back diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & L & \xrightarrow{\mathrm{id}_L} & L & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & U & \longrightarrow & C'_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}_X} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $X \in \widehat{\mathrm{Add}(C)}_n$, and C is finitely presented, we can see that $X \in C^{\perp}$. As $C'_0 \in \mathrm{Add}(C)$, we have $\mathrm{Ext}^1_R(C'_0, X) = 0$. So, $U \cong X \oplus C'_0$. We then have the following pull-back diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & \xrightarrow{\mathrm{id}_K} & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & V & \longrightarrow & C_0 \oplus C'_0 & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_Y} & & \\
0 & \longrightarrow & L & \longrightarrow & X \oplus C'_0 & \longrightarrow & Y & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since K and L are in $\widehat{\mathrm{Add}(C)}_{n-1}$, by the induction hypothesis, we get $V \in \widehat{\mathrm{Add}(C)}_{n-1}$, and so $Y \in \widehat{\mathrm{Add}(C)}_n$. $\square$

The subsequent result bears analogy to [30, Lemma 2.2], and the proof follows a similar approach.

Lemma 3.7. *Let C_R be a finitely presented self-orthogonal R-module, and $m, n \in \mathbb{N}$. Let M_R be an R-module such that there is an exact sequence*

$$0 \to M \to N_1 \to \cdots \to N_m \to Z \to 0$$

for some R-module Z_R and $N_i \in \widehat{\mathrm{Add}(C)}_n$. Then there is an exact sequence

$$0 \to U \to V \to M \to 0$$

for some $U \in \widehat{\mathrm{Add}(C)}_{n-1}$, and some R-module V_R such that there is an exact sequence

$$0 \to V \to C_1 \to \cdots \to C_m \to Z \to 0$$

with each $C_i \in \mathrm{Add}(C)$.

Proof. We prove the statement by induction on m. In case $m = 1$, we have an exact sequence $0 \to M \to N_1 \to Z \to 0$ with $N_1 \in \widehat{\mathrm{Add}(C)}_n$. Since $N_1 \in \widehat{\mathrm{Add}(C)}_n$, there is an exact sequence $0 \to U \to C_1 \to N_1 \to 0$ with $C_1 \in \mathrm{Add}(C)$ and $U \in \widehat{\mathrm{Add}(C)}_{n-1}$. Then, we can construct the following pull-back diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & U & \xrightarrow{\mathrm{id}_U} & U & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & V & \longrightarrow & C_1 & \longrightarrow & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_Z} & & \\
0 & \longrightarrow & M & \longrightarrow & N_1 & \longrightarrow & Z & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Clearly, the left column is just the desired exact sequence.

Next, assume that $m > 1$ and the conclusion holds for $m - 1$. Let M' denote the right R-module $\mathrm{Coker}(M \to N_1)$. Then, by the induction assumption, there is an exact sequence $0 \to U' \to V' \to M' \to 0$ for some $U' \in \widehat{\mathrm{Add}(C)}_{n-1}$ and some V' such that there is an exact sequence

$$0 \to V' \to C_1' \to \cdots \to C_{m-1}' \to Z \to 0$$

with each $C'_i \in \mathrm{Add}(C)$. Now, we can construct the following pull-back diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M & \xrightarrow{\mathrm{id}_M} & M & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U' & \longrightarrow & Y & \longrightarrow & N_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}_{U'}} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U' & \longrightarrow & V' & \longrightarrow & M' & \longrightarrow & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since N_1 and U' are in $\widehat{\mathrm{Add}(C)}_n$, Lemma 3.6 yields that $Y \in \widehat{\mathrm{Add}(C)}_n$. So, there is an exact sequence $0 \to U \to X \to Y \to 0$ with $X \in \mathrm{Add}(C)$ and $U \in \widehat{\mathrm{Add}(C)}_{n-1}$. We then have the following pull-back diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}_{U}} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & V' & \xrightarrow{\mathrm{id}_{V'}} & V' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

It is easy to see that the top row is just the desired sequence. □

Corollary 3.8. *Let $n \in \mathbb{N}$ and C_R be a cotilting R-module with $\mathrm{id}(C_R) = n$. For any R-module M_R, there exists an exact sequence $0 \to U \to V \to M \to 0$ such that $U \in \widehat{\mathrm{Add}(C)}_{n-1}$ and V fits into an exact sequence*

$$0 \to V \to C_0 \to C_1 \to \cdots \to C_{n-1} \to \mho_n(M) \to 0,$$

where each $C_i \in \mathrm{Add}(C)$.

Proof. We consider an exact sequence

$$0 \to M \to E^0 \to \cdots \to E^{n-1} \to \mho_n(M) \to 0$$

in which each E^i is an injective right R-module. Lemma 3.5 yields that $E^i \in \widehat{\mathrm{Add}(C)}_n$. Now, the conclusion follows by Lemma 3.7. □

Lemma 3.9. *Let C_R be a cotilting R-module with $\mathrm{id}(C_R) = n$, and M_R an R-module. If $M_R \in C^{\perp}$, then $\mho_n(M) \in I^{\perp}$ for every injective R-module I_R.*

Proof. Let I_R be an injective R-module. By Lemma 2.1(i), we have

$$\mathrm{Ext}^i_R(I, \mho_n(M)) \cong \mathrm{Ext}^{i+n}_R(I, M)$$

for all $i \geqslant 1$. By Lemma 3.5, we can consider an exact sequence

$$0 \to C_n \to \cdots \to C_1 \to C_0 \to I \to 0$$

with $C_i \in \mathrm{Add}(C)$. It is easy to see that $\mathrm{Add}(C) \subseteq {}^{\perp}M$. So, by Lemma 2.1(ii), we obtain

$$\mathrm{Ext}^{i+n}_R(I, M) \cong \mathrm{Ext}^i_R(C_n, M) = 0$$

for all $i \geqslant 1$. Hence, $\mathrm{Ext}^i_R(I, \mho_n(M)) = 0$ for all $i \geqslant 1$. □

Lemma 3.10. *Let C_R be a cotilting R-module with $\mathrm{id}(C_R) = n$, and let $S = \mathrm{End}(C_R)$. Assume that ${}_SC$ is Artinian. Then, for any self-orthogonal R-module M_R in $C^{\perp} \cap {}^{\perp}C$, the right R-module $\mho_n(M)$ is also self-orthogonal.*

Proof. Since $\mho_0(M) = M$, we may and do assume that $n \geqslant 1$. By Lemma 2.1(i), we have

$$\mathrm{Ext}^i_R(\mho_n(M), \mho_n(M)) \cong \mathrm{Ext}^{i+n}_R(\mho_n(M), M) \tag{3.4}$$

for all $i \geqslant 1$.

According to Corollary 3.8(a), there exists an exact sequence $0 \to U \to V \to M \to 0$ (†) such that $U \in \widehat{\mathrm{Add}(C)}_{n-1}$ and V satisfies an exact sequence

$$0 \to V \to C_0 \to C_1 \to \cdots \to C_{n-1} \to \mho_n(M) \to 0 \ (‡)$$

with $C_i \in \mathrm{Add}(C)$. Given that $M_R \in C^{\perp}$, it follows that $\mathrm{Add}(C) \subseteq {}^{\perp}M$. Applying Lemma 2.1(ii) to the exact sequence (‡) give us

$$\mathrm{Ext}^{i+n}_R(\mho_n(M), M) \cong \mathrm{Ext}^i_R(V, M) \tag{3.5}$$

for all $i \geqslant 1$.

Since M is self-orthogonal, applying the functor $\mathrm{Hom}_R(-, M)$ to (†) leads to the following isomorphism

$$\mathrm{Ext}^i_R(V, M) \cong \mathrm{Ext}^i_R(U, M) \tag{3.6}$$

for all $i \geqslant 1$.

Because C_R is self-orthogonal, it follows that $\mathrm{Add}(C) \subseteq {}^{\perp}C$. Hence, applying Lemma 2.1(ii) to the exact sequence (‡), and noting that $\mathrm{id}(C_R) = n$, we deduce that

$$\mathrm{Ext}^i_R(V, C) \cong \mathrm{Ext}^{i+n}_R(\mho_n(M), C) = 0$$

for all $i \geqslant 1$. Thus, $V \in {}^{\perp}C$. Since $M \in {}^{\perp}C$, the exact sequence (†) yields that $U \in {}^{\perp}C$. On the other hand, $U \in \widehat{\mathrm{Add}(C)}_{n-1}$. Since ${}_SC$ is Artinian, by [17, Corollary 4.4], the R-module C_R is product-complete. Hence, by Lemma 3.3(ii), $U \in \mathrm{Add}(C)$. Consequently, $\mathrm{Ext}^i_R(U, M) = 0$ for all $i \geqslant 1$. Therefore, by (3.4), (3.5) and (3.6), we conclude that $\mathrm{Ext}^i_R(\mho_n(M), \mho_n(M)) = 0$ for all $i \geqslant 1$. □

We recall the following definition from [16]. See also [27].

Definition 3.11. A ring R is said to be *right co-Noetherian* if the injective hull of every simple right R-module is Artinian.

There is a rich source of right co-Noetherian rings as illustrated by the following example:

Example 3.12.
- (i) Matlis [20] proved that the injective hull of every simple module over a commutative Noetherian ring is Artinian, thus every commutative Noetherian ring is right co-Noetherian.
- (ii) By [12, Corollary 2.3], any Noetherian algebra over a commutative Noetherian ring is a right co-Noetherian ring. In particular, the group ring $\mathbb{Z}[G]$ of a finite group G is right co-Noetherian.
- (iii) A ring R is called right V-ring if every simple right R-module is injective. Clearly, every right V-ring is right co-Noetherian.
- (iv) Kaplansky's well-known criterion for commutative von Neumann regular rings implies that a commutative ring is a V-ring if and only if it is von Neumann regular. Consequently, any infinite product of fields is a commutative right co-Noetherian ring that is not Noetherian.
- (v) By [21, Theorem 2.5], the property of being a right V-ring is Morita invariant. Hence, any matrix ring over a right V-ring is again a right V-ring, yielding numerous examples of non-commutative right co-Noetherian rings.
- (vi) A right Noetherian ring is not necessarily right co-Noetherian. Indeed, there exists a left and right Artinian ring that is not right co-Noetherian; see [1, Exercise 24.9, p. 286].

Theorem 3.13. *Let R be a right Artinian ring, let C_R be a cotilting R-module, and set $S = \mathrm{End}(C_R)$.*

1. *If S satisfies (ARC), then R^{op} satisfies (DARC).*
2. *The converse holds under the additional assumptions that R is right co-Noetherian and S is left Artinian.*

Proof. (1) Assume that S satisfies (ARC). We prove that $R^{\rm op}$ satisfies (DARC). Let M_R be an Artinian R-module such that $\mathrm{Ext}^i_R(M\oplus I, M)=0$ for every injective R-module I_R and all $i>0$. We show that M_R is injective.

Because R is right Artinian and M_R is Artinian, [10, Corollary 2.3.24] implies that M_R is finitely generated.

Let $\mathrm{id}({}_SC)=n$. By [13, Theorem 2.7], we have $n=\mathrm{id}(C_R)$. By [14, Theorem 1.2], there exists an exact sequence

$$0\to U\to V\to M\to 0 \tag{3.7}$$

with $V\in {}^{\perp}C\bigcap \text{mod-}R$ and $U\in \widehat{\mathrm{add}(C)}_{n-1}$.

Since $\mathrm{id}(C_R)<\infty$ and $U\in\widehat{\mathrm{add}(C)}_{n-1}$, we have $\mathrm{id}(U_R)<\infty$. From $M\in I^{\perp}$ for all injective I_R, it follows that $M\in X^{\perp}$ for every R-module X_R of finite injective dimension. Hence, $M\in C^{\perp}\bigcap\text{mod-}R$ and $M\in U^{\perp}\bigcap\text{mod-}R$. Since $U\in\widehat{\mathrm{add}(C)}_{n-1}$, we also have $U\in C^{\perp}\bigcap\text{mod-}R$, so $V\in C^{\perp}\bigcap\text{mod-}R$ by (3.7).

Applying $\mathrm{Hom}(-,M)$ to (3.7) yields

$$\cdots\to\mathrm{Ext}^i_R(M,M)\to\mathrm{Ext}^i_R(V,M)\to\mathrm{Ext}^i_R(U,M)\to\cdots.$$

As M_R is self-orthogonal, we get $\mathrm{Ext}^i_R(V,M)=0$ for all $i>0$. Since $V\in {}^{\perp}C\bigcap\text{mod-}R$ and $U\in\widehat{\mathrm{add}(C)}_{n-1}$, we also have $\mathrm{Ext}^i_R(V,U)=0$ for all $i>0$. Applying $\mathrm{Hom}_R(V,-)$ to (3.7) gives $\mathrm{Ext}^i_R(V,V)=0$ for all $i>0$, hence $V\in (V\oplus C)^{\perp}\bigcap {}^{\perp}C\bigcap\text{mod-}R$.

Set $N=\mathrm{Hom}_R(V,C)$. Lemma 3.1 gives

$$\begin{aligned}\mathrm{Ext}^i_S(N,N\oplus S)&=\mathrm{Ext}^i_S(\mathrm{Hom}_R(V,C),\mathrm{Hom}_R(V,C)\oplus\mathrm{Hom}_R(C,C))\\&\cong\mathrm{Ext}^i_R(V\oplus C,V)\\&=0\end{aligned}$$

for all $i>0$. By assumption, N is finitely generated projective over S, and applying Lemma 3.1 again gives $V\in\mathrm{add}(C)$. Then, from (3.7), we have $M\in\widehat{\mathrm{add}(C)}_n$, so $\mathrm{id}(M_R)<\infty$. Consider an exact sequence $0\to M\to I\to X\to 0$ of right R-modules with I injective and $\mathrm{id}(X_R)<\infty$. Since $\mathrm{Ext}^1_R(X,M)=0$, applying the functor $\mathrm{Hom}_R(X,-)$ to this sequence shows that it splits, and consequently M_R is injective. Therefore, $R^{\rm op}$ satisfies (DARC).

(2) Assume that $R^{\rm op}$ satisfies (DARC). We aim to show that S satisfies (ARC). To this end, let $N\in S$–mod be such that $\mathrm{Ext}^{i\geqslant 1}_S(N,N\oplus S)=0$. As the ring S is left Noetherian and ${}_SN$ is finitely generated, we may choose syzygies of ${}_SN$ to be finitely generated. By Lemma 2.1, for each $i\geqslant 1$, we have

$$\begin{aligned}\mathrm{Ext}^i_S(\Omega^n_SN,\Omega^n_SN)&\cong\mathrm{Ext}^{i+n}_S(N,\Omega^n_SN)\\&\cong\mathrm{Ext}^i_S(N,N)\\&=0,\end{aligned}$$

and

$$\operatorname{Ext}^i_S(\Omega^n_S N, S) \cong \operatorname{Ext}^{i+n}_S(N, S) = 0.$$

Hence, $\operatorname{Ext}^{i\geqslant 1}_S(\Omega^n_S N, \Omega^n_S N \oplus S) = 0$. On the other hand, we know that $\operatorname{id}({}_S C) = n$, which implies $\Omega^n_S N \in {}^{\perp}({}_S C) \cap S$–mod. Thus, by Lemma 3.1, we can express $\Omega^n_S N \cong \operatorname{Hom}_R(M, C)$ for some $M \in {}^{\perp}(C_R) \cap$ mod–R. Since $\operatorname{Hom}_R(C, C) = S$ and $M \oplus C \in {}^{\perp}(C_R) \cap$ mod–R, by Lemma 3.1, we have the following isomorphisms for every $i \geqslant 1$:

$$(3.8)\qquad \begin{aligned} \operatorname{Ext}^i_R(M \oplus C, M) &\cong \operatorname{Ext}^i_S(\operatorname{Hom}_R(M, C), \operatorname{Hom}_R(M, C) \oplus \operatorname{Hom}_R(C, C)) \\ &\cong \operatorname{Ext}^i_S(\Omega^n_S N, \Omega^n_S N \oplus S) \\ &= 0. \end{aligned}$$

In particular, we see that M_R is self-orthogonal and $M \in C^{\perp} \cap {}^{\perp} C$. From Lemmas 3.9 and 3.10, we obtain that

$$(3.9)\qquad \operatorname{Ext}^i_R(\mho_n(M) \oplus I, \mho_n(M)) = 0$$

for every injective R-module I_R and for all $i \geqslant 1$.

As the ring R is right Artinian, the finitely generated R-module M_R is Artinian. Since the R-module M_R is Artinian and R is right co-Noetherian, by applying [26, Theorem 3.21], we can choose $\mho_n(M)$ to be an Artinian right R-module. As R^{op} satisfies (DARC), by (3.9), we induce that $\mho_n(M)$ is injective, leading to $\operatorname{id}(M_R) \leqslant n$. So, we obtain from Lemma 3.4 that $M \in \widetilde{\operatorname{add}(C)}_n$. As $M \in (C_R)^{\perp}$, Lemma 3.3(i) implies that $M \in \operatorname{add}(C)$. So, $\Omega^n_S N \cong \operatorname{Hom}_R(M, C)$ is a projective left S-module. Consequently, $\operatorname{pd}({}_S N) < \infty$. Since $N \in {}^{\perp}S$ too, it is easy to see that ${}_S N$ is a projective S-module. Therefore, S satisfies (ARC). ☐

Remark 3.14. It is well known that (ARC) holds for complete intersection local rings. Moreover, by [2, Theorem 3] and [9, Theorem 4.5(1)], the conjecture holds for all Gorenstein local rings if and only if it holds for every Artinian Gorenstein local ring. Consequently, among commutative Noetherian rings, Artinian Gorenstein local rings constitute a natural and fundamental class for studying (ARC). Since an Artin algebra R satisfies (ARC) if and only if the ring R^{op} satisfies (DARC), it follows that, over an Artinian Gorenstein local ring, (ARC) and (DARC) are equivalent.

Although Theorem 3.13 does not provide new instances in which (ARC) is confirmed, it shows that, under certain circumstances, the conjecture is equivalent to (DARC).

We next give an example that provides a broad class of rings satisfying (ARC) by Theorem 3.13. To present it, we first recall the definition of quivers and then state a lemma.

Recall that a quiver $\mathcal{Q}$ is a directed graph (V, E), where V and E are respectively the sets of vertices and arrows of $\mathcal{Q}$, endowed with a pair of functions s and t which assign to any arrow a of $\mathcal{Q}$ its source and target vertices. If R is a ring and $\mathcal{Q}$ is a quiver, a representation $\mathcal{X}$ of $\mathcal{Q}$ by R-modules and R-homomorphisms is obtained by associating to each vertex v an R-module $\mathcal{X}_v$ and to each arrow $a : v \to w$ an R-homomorphism $\mathcal{X}_a : \mathcal{X}_v \to \mathcal{X}_w$. If $\mathcal{X}$ and $\mathcal{Y}$ are two representations of $\mathcal{Q}$, then a morphism $f : \mathcal{X} \to \mathcal{Y}$ is determined by a family $\{f_v : \mathcal{X}_v \to \mathcal{Y}_v\}_{v\in V}$ of R-homomorphisms such that for any arrow $a : v \to w$, the commutativity condition $\mathcal{Y}_a f_v = f_w \mathcal{X}_a$ holds. For a given quiver $\mathcal{Q}$, the representations of $\mathcal{Q}$ and the morphisms between them form a category, which is denoted by $\mathrm{Rep}(\mathcal{Q}, R)$.

Let $\mathcal{Q}$ be a quiver. Let $R\mathcal{Q}$ be the free R-module with basis the set of all paths in $\mathcal{Q}$, endowed with the following multiplication: if p and p' are paths, let pp' be the concatenation of p and p' provided the tail of p is the head of p', and the zero vector otherwise, and extend this multiplication bilinearly to $R\mathcal{Q}$. It is known that if $\mathcal{Q}$ is finite, that is, if V and E are both finite, then the category $\mathrm{Rep}(\mathcal{Q}, R)$ is equivalent to the category Mod-$R\mathcal{Q}$. Note that this equivalence restricts to an equivalence of categories $\mathrm{rep}(\mathcal{Q}, R)$ and mod-$R\mathcal{Q}$. Here, $\mathrm{rep}(\mathcal{Q}, R)$ denotes the subcategory of $\mathrm{Rep}(\mathcal{Q}, R)$ consisting of all finitely generated representations. The proof of this assertion is similar to the case where R is a field; see [4, Chapter III, Theorem 1.6].

Lemma 3.15. *Let $\mathcal{Q}\colon 1 \to 2$ be a quiver. Then:*

(i) *If R is a right Artinian ring, then $R\mathcal{Q}$ is also a right Artinian ring.*
(ii) *If R is right co-Noetherian, then $R\mathcal{Q}$ is also a right co-Noetherian ring.*
(iii) $R\mathcal{Q} \cong \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}$.
(iv) $(R\mathcal{Q})^{\mathrm{op}} \cong R^{\mathrm{op}}\mathcal{Q}^{\mathrm{op}}$.

Proof. (i) Since R is a right Artinian ring and $R\mathcal{Q}$ is a finitely generated free R-module, it follows immediately that the ring $R\mathcal{Q}$ is also right Artinian.

(ii) Let M be an arbitrary simple $R\mathcal{Q}$-module. We need to show that the injective hull of M is Artinian. The representation of M in $\mathrm{Rep}(\mathcal{Q}, R)$ is either $r_1 = (S \to 0)$ or $r_2 = (0 \to S)$, where S is a simple R-module. On the other hand, by [3, Theorem 3.1], the injective hulls of r_1 and r_2 are given by $E(r_1) = (E(S) \to 0)$ and $E(r_2) = (E(S) \to E(S))$, respectively. Since R is a right co-Noetherian ring, $E(S)$ is an Artinian R-module. Consequently, the injective hull of M is an Artinian $R\mathcal{Q}$-module. Therefore, the ring $R\mathcal{Q}$ is right co-Noetherian.

The proofs of assertions (iii) and (iv) are analogous to the case where R is a field; see [4, Lemma II.1.12] and [4, Exercise II.4.1], respectively. □

Example 3.16. Let Γ be a right co-Noetherian ring, and let E_Γ be a cotilting module. Set $\Lambda = \mathrm{End}(E_\Gamma)$, and assume that Γ is right Artinian and that Λ is left Artinian. Put

$R = \begin{pmatrix} \Gamma & \Gamma \\ 0 & \Gamma \end{pmatrix}$ and $S = \begin{pmatrix} \Lambda & \Lambda \\ 0 & \Lambda \end{pmatrix}$. Set $C = (0 \to E) \oplus (E \to E)$. By Example 2.6(iv), C_R is a cotilting R-module and $S \cong \mathrm{End}(C_R)^{\mathrm{op}}$. By Lemma 3.15, the ring R is right Artinian and right co-Noetherian. Moreover, the ring S is left Artinian. If R^{op} satisfies (DARC), then it follows from Theorem 3.13 that S satisfies (ARC) conjecture.

Acknowledgement

The authors thank the referee for their careful reading of the manuscript and for the valuable and constructive comments, which have significantly improved the paper. In particular, addressing the referee's second comment led to a substantial improvement of the main theorem. They also thank Asghar Daneshvar for introducing the papers [16, 12] and for his insightful comments on this work.

References

[1] F. W. Anderson and K. R. Fuller, *Rings and categories of modules*, 2nd ed., Graduate Texts in Mathematics, **13**, New York: Springer-Verlag, (1992).

[2] T. Araya, *The Auslander-Reiten conjecture for Gorenstein rings*, Proc. Am. Math. Soc., **137**(6), (2009), 1941-1944.

[3] J. Asadollahi, R. Hafezi and M. H. Keshavarz, *Minimal injective resolutions and Auslander-Gorenstein property for path algebras*, Commun. Algebra, **45**(6), (2017), 2557-2568.

[4] I. Assem, D. Simson, A. Skowroński, *Elements of representation theory of associative algebras*, vol. 1, Techniques of representation theory, London Mathematical Society Student Texts, **65**, Cambridge University Press, Cambridge, (2006).

[5] M. Auslander, *Functors and morphisms determined by objects*, Represent. Theory of Algebras, Proc. Phila. Conf., Lect. Notes pure appl. Math., **37**, (1978), 1-244.

[6] M. Auslander and I. Reiten, *On a generalized version of the Nakayama conjecture*, Proc. Amer. Math. Soc., **52**, (1975), 69-74.

[7] M. Auslander and I. Reiten, *Applications of contravariantly finite subcategories*, Adv. Math., **86**(1), (1991), 111-152.

[8] M. Auslander and I. Reiten, *Cohen-Macaulay and Gorenstein Artin algebras*, Representation theory of finite groups and finite-dimensional algebras, Proc. Conf., Bielefeld/Ger. 1991, Prog. Math., **95**, (1991), 221-245.

[9] O. Celikbas and R. Takahashi, *Auslander-Reiten conjecture and Auslander-Reiten duality*, J. Algebra, **382**, (2013), 100-114.

[10] E. E. Enochs and O. M. G. Jenda, *Relative homological algebra*, Walter de Gruyter, Berlin-New York, (2000).

[11] H. Eshraghi and A. Mahin Fallah, *Auslander-Reiten conjecture in a dual vein*, Algebr Represent Theory, **24**(5), (2021), 1279-1294.

[12] Y. Hirano, *On injective hulls of simple modules*, J. Algebra, **225**(1), (2000), 299-308.

[13] Z. Y. Huang, *Generalized tilting modules with finite injective dimension*, J. Algebra, **311**(2), (2007), 619-634.

[14] Z. Y. Huang, *Selforthogonal modules with finite injective dimension III*, Algebr. Represent. Theory, **12**(2-5), (2009), 371-384.

[15] Z. Y. Huang and G. Tang, *Self-orthogonal modules over coherent rings*, Pure Appl. Algebra, **161**(1-2), (2001), 167-176.

[16] J. P. Jans, *On co-Noetherian rings*, J. London Math. Soc., **s2-1**(1), (1969), 588-590.

[17] H. Krause and M. Saorín, *On minimal approximations of modules*, in: E. L. Green, B. Huisgen-Zimmermann (Eds.), Trends in the Representation Theory of Finite Dimensional Algebras, in: Contemp. Math., **229**, (1998), 227–236.

[18] A. Mahin Fallah, *Auslander-Reiten conjecture of the tilting theorem*, J. Algebra Appl, **23**(1), (2024), 2450013.

[19] F. Mantese and I. Reiten, *Wakamatsu tilting modules*, J. Algebra, **278**(2), (2004), 532-552.

[20] E. Matlis, *Modules with descending chain condition*, Trans. Amer. Math. Soc, **97**(3), (1960), 495-508.

[21] G. O. Michler and O. E. Villamayor, *On rings whose simple modules are injective*, J. Algebra, **25**(1), (1973), 185-201.

[22] J. Miyachi, *Cohen-Macaulay approximations and Noetherian algebras*, Commun. Algebra, **26**(7), (1998), 2181-2190.

[23] J. Miyachi, *Duality for derived categories and cotilting bimodules*, J. Algebra, **185**(2), (1996), 583-603.

[24] Y. Miyashita, *Tilting modules of finite projective dimension*, Math. Z., **193**(1), (1986), 113-146.

[25] T. Nakayama, *On algebras with complete homology*, Abh. Math. Sem. Univ. Hamburg, **22**, (1958), 300-307.

[26] D. W. Sharpe and P. Vamos, *Injective modules*, Cambridge Tracts in Mathematics and mathematical Physics, **62**, Cambridge Uiversity Press, Cambridge (1972).

[27] R. L. Snider, *Injective hulls of simple modules over group rings*, in “Proceedings of the Ohio University Ring Theory Conference" **70**, 223-226, Dekker, New York, (1977).

[28] T. Wakamatsu, *Tilting modules and Auslander's Gorenstein property*, J. Algebra, **275**(1), (2004), 3-39.

[29] J. Wei, *Tilting invariance of the Auslander-Reiten conjecture*, Math. Res. Lett., **17**(1), (2010), 171-176.

[30] J. Wei and C. Xi, *A characterization of the tilting pair*, J. Algebra, **317**(1), (2007), 376-391.

[31] P. Zhang, *Monomorphism categories, cotilting theory, and Gorenstein-projective modules*, J. Algebra, **339**(1), (2011), 181-202.

K. Divaani-Aazar, Department of Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran, Iran.

Email address: kdivaani@ipm.ir

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395-5746, Tehran, Iran.

Email address: amfallah@ipm.ir, ali.mahinfallah@gmail.com

Tousi, Department of Mathematics, Faculty of Mathematical Sciences, Shahid Beheshti University, Tehran, Iran.

Email address: mtousi@ipm.ir