FRACTIONAL BESSEL-SOBOLEV AND BESSEL B-POTENTIAL SPACES

M. CHEGAAR AND Á. P. HORVÁTH

Abstract. After proving the equivalence of the Bessel K-functional and the corresponding spherical modulus of smoothness we define fractional Bessel-Sobolev spaces. As an analog of the classical one the imbedding relation of fractional Bessel-Sobolev and Bessel B-potential spaces is pointed out. Applying the defined notions a potential theoretic characterization of removable sets with respect to a certain fractional partial differential operator is derived.

1. Introduction

Bessel and fractional Bessel differential operators, i.e. operators related to

$$\Delta_a := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha_i + 1}{x_i}\frac{\partial}{\partial x_i}, \ \ \alpha_i > -\frac{1}{2},$$

(where $a = (a_1, \dots, a_n)$, $a_i = 2\alpha_i+1$; cf. (1) and (2)) occur frequently in physics and in practice. Without claiming to be exhaustive, we list some recent works involving Bessel operators. For example, in [25] the Laplace transform homotopy analysis method is applied to solve numerically a fractional-order telegraph equation with a Bessel operator. In [3] the author investigates the Bessel heat equation and proves the existence and uniqueness of the extremal function associated with the gaussian transformation. In [30] generalized Bessel potential is studied and is considered as a solution to the singular screened Poisson equation.

The generalized (or Bessel) translation defined by Delsarte, [12] and developed by Levitan, [21] has proven to be a fairly effective tool for examining these operators. Application of Bessel translation rather than the standard one modifies the geometry of the space appropriately. This is why it is worth to introduce the corresponding modified spaces. For example, [18] refines the theory of Hardy spaces for certain Bessel operators by introducing atomic decompositions to study singular integrals and related operators. In [16] Riesz potentials are investigated in Morrey- and BMO spaces generated by Laplace-Bessel differential operators. Generalized potential spaces were examined e.g. to find perfect functional completion of certain class of functions, see [31], or to study the properties of the corresponding capacity, see [29].

The study of function spaces has long been central in analysis. These spaces provide the right framework for formulating and solving problems in partial differential equations, harmonic analysis, and approximation theory. For instance, to study the convergence properties of a numerical solution, to characterize removable

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sets or to develop the suitable spectral theory, the investigation of the underlying space is essential. Fractional order smoothness was first considered by Gagliardo, and later by Nikol'skii and Besov. These scales capture more delicate properties of functions than classical Sobolev spaces.

First we focus on the equivalence of modulus of smoothness and the corresponding K-functional. It is a crucial inequality in approximation theory and also in the theory of interpolation of spaces. In [15] several results on K-functional are summarized in classical cases. In the following, we extend the one-dimensional results of [28] to higher dimensions. The main tool of this section is the weighted spherical modulus of smoothness introduced in [11].

This equivalence allows to define the appropriate spherical Gagliardo-type seminorm (cf.[26]) and fractional Bessel-Sobolev spaces, i.e. to extend the results of [28] and [29] to fractional exponents.

As usual, Bessel translation defines a convolution and so Bessel B-potential spaces, i.e. the spaces of functions derived by Bessel convolution with the Bessel kernel. Classical Bessel potential spaces were introduced systematically in a series of papers by Aronszajn and Smith, see e.g. [4] and by Calderón, see [9] for the one-dimensional case. The analogy of Calderón's theorem is derived in the Bessel case, see [29] or [19]. The fractional case is a little different, cf.[1]. By examining the defined norms, a suitable imbedding theorem can be proven.

As an application, we characterize removable sets with respect to certain fractional partial differential operators. The potential theoretic characterization of removable sets is due to Carleson, see [10]. For Laplace-Bessel type operators similar results can be found in [19]. For fractional parabolic operators the results of [24] have been extended to both classical and Bessel cases in [11]. In the last section we give a characterization of removable sets with respect to fractional elliptic operators via capacity defined by fractional Bessel B-potential spaces.

2. Notation, preliminaries

In this preparatory section we fix notation and collect the necessary tools and notions.

2.1. **Norms, spaces.** On $\mathbb{R}^n_+ := \{(x_1, \dots, x_n) : x_i > 0 \ i = 1, \dots, n\}$ we introduce the $L^p_a(\mathbb{R}^n_+)$ space, $1 \le p < \infty$ with norm

$$\|f\|^p_{p,a} = \int_{\mathbb{R}^n_+} |f(x)|^p x^a dx,$$

where $a = (a_1, \dots, a_n)$, $a_i > 0$, $i = 1, \dots, n$; $x^a = \prod_{i=1}^n x_i^{a_i}$. For $p = \infty$ we use the standard $L^\infty(\mathbb{R}^n_+)$ space.
The appropriate weighted inner product is as follows.

$$\langle f, g\rangle_a := \int_{\mathbb{R}^n_+} f(x)g(x)x^a dx.$$

$\mathcal{S}_e$ stands for the even functions from the Schwartz class, i.e.
$\mathcal{S}_e := \left\{ f \in \mathcal{S} : \frac{\partial^{2m+1}}{\partial x_i^{2m+1}} f|_{x_i=0} = 0, \ \forall m \in \mathbb{N}, \ i = 1, \dots, n \right\}.$
If $h \in \mathcal{S}'_e$, $\varphi \in \mathcal{S}_e$ we denote the effect of h on φ by $\langle h, \varphi\rangle_a$ as well.

2.2. **Bessel differential operators.** The Bessel-Laplace operator on $\mathbb{R}^n_+$ is defined as follows.

$$\Delta_a := \sum_{i=1}^{n} B_{\alpha_i}, \text{ where } B_{\alpha_i} := \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha_i + 1}{x_i}\frac{\partial}{\partial x_i}, \tag{1}$$

and

$$a_i = 2\alpha_i + 1, \tag{2}$$

(i.e. $\alpha_i > -\frac{1}{2}$, $i = 1, \ldots, n$). Hereafter we deal with partial differential operators which contain integer or fractional powers of Δ_a or $I - \Delta_a$.

2.3. **Bessel functions, Bessel translation.** The entire Bessel functions are

$$j_\alpha(z) = \Gamma(\alpha+1)\left(\frac{2}{z}\right)^\alpha J_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(k+\alpha+1)}\left(\frac{z}{2}\right)^{2k}. \tag{3}$$

In this paper we assume that $\alpha > -\frac{1}{2}$. We note that $j_{-\frac{1}{2}}(x) = \cos x$. Bessel functions $j_\alpha(\lambda x)$, are the eigenfunctions of B_α, i.e.

$$B_\alpha j_\alpha(\lambda x) = -\lambda^2 j_\alpha(\lambda x),$$

see [21]. Therefore, we call the operator Bessel and use the notation $a_i = 2\alpha_i + 1$ in parallel. Entire Bessel functions are even and bounded, and their supremum is taken at zero:

$$\|j_\alpha\|_{\infty,\mathbb{R}_+} = j_\alpha(0) = 1. \tag{4}$$

Below we need the derivatives of the Bessel functions.

$$j'_\alpha(z) = -\frac{1}{2(\alpha+1)} z j_{\alpha+1}(z), \tag{5}$$

see [6].

We introduce the following abbreviation, cf. [32].

$$\daleth_a(x,\xi) := \prod_{i=1}^{n} j_{\alpha_i}(x_i\xi_i).$$

Subsequently we also need the modified Bessel function of the second kind K_ν.

$$K_\nu(x) = \frac{\pi}{2}\frac{i^\nu J_{-\nu}(ix) - i^{-\nu}J_\nu(ix)}{\sin\nu\pi}, \tag{6}$$

see [6]. Growth properties of K_ν are as follows.
Around zero:

$$K_\nu(r) \sim \begin{cases} -\ln\frac{r}{2} - c, \text{ if } \nu = 0 \\ C(\nu)r^{-\nu}, \text{ if } \nu > 0, \end{cases} \tag{7}$$

Around infinity:

$$K_\nu(r) \sim \frac{c}{\sqrt{r}}e^{-r}, \tag{8}$$

see e.g. [2, (1.2.20)-(1.2.22)].
Moreover around zero for $\nu \geq 0$ we have

$$|(r^\nu K_\nu(r))'| \leq c \begin{cases} r^{2\nu-1}\ln\frac{1}{r}, \text{ if } 1 > 2\nu, \ \nu \in \mathbb{N} \\ r^{2\nu-1}, \text{ if } 1 > 2\nu, \ \nu \notin \mathbb{N} \\ 1, \text{ if } 1 < 2\nu, \ \nu \notin \mathbb{N} \\ r, \text{ if } 1 < 2\nu, \ \nu \notin \mathbb{N}, \end{cases}, \tag{9}$$

see [13, (40),(41)].

We introduce the Bessel or generalized translation of Delsarte. With the above notation the Bessel translation of a function, f (see e.g. [21], [28], [32]) is

$$T_a^t f(x) = T_{a_n}^{t_n} \dots T_{a_1}^{t_1} f(x_1, \dots, x_n),$$

where

$$T_{a_i}^{t_i} f(x_1, \dots, x_n)$$

(10)

$$= \frac{\Gamma(\alpha_i + 1)}{\sqrt{\pi}\Gamma\left(\alpha_i + \frac{1}{2}\right)} \int_0^\pi f(x_1, \dots, \sqrt{x_i^2 + t_i^2 - 2x_i t_i \cos\vartheta_i}, x_{i+1}, \dots, x_n) \sin\vartheta^{2\alpha_i} d\vartheta_i.$$

The definition immediately implies symmetry,

$$T_a^t f(x) = T_a^x f(t).$$

Succession of two translations is symmetric as well.

$$T_a^y f(x) T_a^z f(x) = T_a^z f(x) T_a^y f(x), \tag{11}$$

see [21, (7.1)].
It is also an immediate consequence of the definition that T_a is positive operator. Bessel translation is bounded in L_a^p, i.e. for all a $(a_i > 0)$,

$$\|T_a^t f(x)\|_{p,a} \le \|f\|_{p,a}, \quad 1 \le p \le \infty, \tag{12}$$

see e.g. [21].

2.4. **Bessel convolution, Hankel transform.** The generalized convolution with respect to the Bessel translation is

$$f *_a g = \int_{\mathbb{R}_+^n} T_{a,x}^t f(x) g(x) x^a dx. \tag{13}$$

Bessel convolution possesses all the main properties of the standard one, i.e.

$$f *_a g = g *_a f, \quad f *_a (g *_a h) = (f *_a g) *_a h. \tag{14}$$

Furthermore, Young's inequality also holds, i. e. if $1 \le p, q, r \le \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$; if $f \in L_a^p$ and $g \in L_a^q$, then

$$\|f *_a g\|_{r,a} \le \|f\|_{p,a} \|g\|_{q,a}, \tag{15}$$

see [32, (3.178)].

The integral transformation that corresponds to the Bessel translation and convolution is the Hankel transform. The Hankel transform $\mathcal{H}_a$ of f is a function of ξ defined by

$$\mathcal{H}_a(f, \xi) = \hat{f}(\xi) = \int_{\mathbb{R}_+^n} f(x) \daleth_a(x, \xi) x^a dx.$$

If $f \in L_a^1(\mathbb{R}_+^n)$ is of bounded variation in a neighborhood of a point x of continuity of f, then the next inversion formula holds, see [page 38.][32].

$$f(x) = \mathcal{H}_a^{-1}(\hat{f}(\xi))(x) = \frac{2^{n-|a|}}{\prod_{i=1}^n \Gamma^2(\alpha_i + 1)} \int_{\mathbb{R}_+^n} \hat{f}(\xi) \daleth_a(x, \xi) \xi^a d\xi. \tag{16}$$

The relationship of Hankel transform to translation and convolution is as follows.

$$\mathcal{H}_a(T_a^y f)(\xi) = \daleth_a(\xi, y) \mathcal{H}_a(f)(\xi); \quad T_a^y(\mathcal{H}_a f)(\xi) = \mathcal{H}_a(\daleth_a(x, y) f(x)), \tag{17}$$

see [32, (3.158)].The following property shows that the effect of the Hankel transform on the Bessel convolution coincides with the effect of the Fourier transform on the standard convolution, which is crucial in these studies. For $f, g \in \mathcal{S}_e$ we have

$$\mathcal{H}_a(f *_a g) = \hat{f}\hat{g}, \tag{18}$$

see e.g. [32, (3.176)].

Subsequently we need the Hankel transform of a radial function. We start with the following formula. Let S^n_{1+} be the positive part of the n-dimensional unite sphere.

$$\frac{1}{|S^n_{1+}|_a} \int_{S^n_{1+}} \daleth_a(r\Theta, \xi))\Theta^a dS(\Theta) = j_{\frac{n+|a|}{2}-1}(r|\xi|), \tag{19}$$

where

$$|S^n_{1+}|_a = \int_{S^n_{1+}} \Theta^a dS(\Theta) = \frac{\prod_{i=1}^n \Gamma(\alpha_i + 1)}{2^{n-1}\Gamma\left(\frac{n+|a|}{2}\right)}, \tag{20}$$

and dS stands for the integration on the sphere, cf.[32, (3.140)].

Then let f be a radial function, i.e. $f(x) = \varphi(|x|)$, where φ is a function of one variable. Denoting by $r = |x|$, $\mathcal{H}_a(f)$ is also radial, furthermore we have

$$\mathcal{H}_a(f)(\xi) = |S^n_{1+}|_a \int_0^\infty \varphi(r) j_{\frac{n+|a|}{2}-1}(|\xi|r) r^{n+|a|-1} dr, \tag{21}$$

see [30, Lemma 3.2].

2.5. **Differences, moduli of smoothness.** The difference induced by Bessel translation is as follows.

$$\Delta_{y,a} f(x) := f(x) - T^y_a f(x).$$

Since in the standard case, i.e. $\Delta_y f(x) = f(x) - f(x+t)$, translation has the group property, it can be iterated as $\Delta^k_y f(x) = \sum_{l=0}^k (-1)^l \binom{k}{l} f(x+lt) = \Delta_y \Delta^{k-1}_y f(x)$. Unlike the standard case, Bessel translations do not form an additive group (cf. (10)), so they can be iterated in two ways.

Notation.

The iterated difference of a function f is

$$\Delta^k_{y,a} f(x) := \Delta_{y,a}\Delta^{k-1}_{y,a} f(x), \tag{22}$$

and the modified iterated difference is

$$\tilde{\Delta}^k_{y,a} f(x) := \sum_{l=0}^k (-1)^l \binom{k}{l} T^{\sqrt{l}y}_a f(x). \tag{23}$$

$\Delta^k_{y,a}$ is used to state the main theorem of the next section, and to prove it, we need both iterations.

In view of (17) we introduce the following notation.

Notation.

$$\mathcal{J}_{\alpha,k}(x) := \sum_{l=0}^k (-1)^l \binom{k}{l} j_\alpha(\sqrt{l}x).$$

Subsequently the following property of $\mathcal{J}_{\alpha,k}$ will be useful.

Lemma 1. [28, Lemma 4.1]

$$\mathcal{H}_\alpha^{-1}\left(\frac{\mathcal{J}_{\alpha,k}(\xi)}{\xi^{2k}}\right) \in L^1_\alpha(\mathbb{R}_+). \tag{24}$$

These differences lead to the corresponding spherical differences and moduli of smoothness.

Definition 1. *The spherical and iterated spherical differences are as follows.*

$$\Delta_{sph,r,a}f(x) := \frac{1}{|S^n_{1+}|_a}\int_{S^n_{1+}} \Delta_{r\Theta,a}f(x)\Theta^a dS(\Theta);$$

$$\Delta^k_{sph,r,a}f(x) := \Delta_{sph,r,a}\Delta^{k-1}_{sph,r,a}f(x),$$

and

$$\tilde{\Delta}^k_{sph,r,a}f(x) := \frac{1}{|S^n_{1+}|_a}\int_{S^n_{1+}} \tilde{\Delta}^k_{r\Theta,a}f(x)\Theta^a dS(\Theta),$$

cf.(20)*. Let* $1 \le p \le \infty$*. The spherical Bessel p-modulus of smoothness is*

$$\omega_{sph,m}(f,t)_{a,p} := \sup_{0<h<t}\|\Delta^m_{sph,h,a}f\|_{p,a}.$$

The first version of modulus of smoothness (modulus of continuity), $\omega(f,\delta) := \sup_{|x-y|<\delta}|f(x)-f(y)|$, was introduced to study the relation of the properties of a continuous function to its best approximation with a polynomial of bounded degree on a finite interval of the real line (Jackson and Bernstein type theorems). Then it turned out that this is also closely related to the so-called K-functional, see the next section. Now we list some elementary properties of the defined spherical modulus, which are direct consequences of the definition.

Properties.

In view of (17) and (19) we have

$$\mathcal{H}_a(\Delta_{sph,r,a}f)(\xi) = (1 - j_{\frac{n+|a|}{2}-1}(r|\xi|))(\mathcal{H}_af)(\xi). \tag{25}$$

From the definition and from (12) it follows that $\omega_{sph,m}(f,t)_{a,p}$ is increasing in t, moreover

$$\omega_{sph,m}(f+g,t)_{a,p} \le \omega_{sph,m}(f,t)_{a,p} + \omega_{sph,m}(g,t)_{a,p} \tag{26}$$

and

$$\omega_{sph,m}(f,\delta)_{a,p} \le 2^m\|f\|_{p,a}. \tag{27}$$

The spherical difference was introduced in [11]. The main point of that study was to demonstrate the relationship between Laplace and Bessel-Laplace type operators, which is a consequence of the similarity between the Fourier and Hankel transforms of radial functions. In [11], these results were applied to describe removable sets with respect to certain parabolic-type fractional equations. It turned out that the obstacle to achieving sharp results is the unexplored background of the topic, which is also the motivation for this work.

3. K-functional

Peetre's K-functional plays a decisive role for instance in the proofs of Marchaud-type and reverse Marchaud-type, sharp Ul'yanov and Kolyada-type inequalities which are fundamental in approximation theory, cf. e.g. [15]. A nice application of the equivalence of certain generalized K-functionals and moduli of smoothness e.g. is an approximation process by families of generalized sampling series, see [5].

In this section we give the definition of the Bessel-Sobolev space and the corresponding K-functional, see [29], [19] and in 1-dimension see [28, (1.9)]. The following theorem describes the connection between the defined K-functional and spherical modulus of smoothness.

Hankel transformation is proven to be a useful tool to derive this result. According to (17), the Hankel transform of a shifted function can be expressed as the Hankel transform of the function multiplied by an appropriate Bessel function. This motivates the introduction of spherical modulus of smoothness. Indeed, the formula for the Hankel transform of a radial function together with Lemma 1 imply that $\frac{\mathcal{J}_{\frac{n+|a|}{2}-1,m}(|\xi|)}{|\xi|^{2m}}$ is a Hankel multiplier, cf. Lemma 4, which is a crucial fact in the proof of the main theorem of this section.

Although in Section 4 we use the results of this section to define the spherical Gagliardo seminorm and fractional Bessel-Sobolev spaces, the last Corollary of this section as a direct consequence of Theorem 1 is also interesting in itself from an approximation theory aspect.

In accordance with the one-dimensional case (cf. [28], see also [19]), we introduce Bessel-Sobolev spaces with integer parameters.

Definition 2. *Let $1 \le p \le \infty$; $m \in \mathbb{N}$. The Bessel-Sobolev space on $\mathbb{R}^n_+$ is*

$$W^{m,p}_{\Delta_a} := \left\{ f : \mathbb{R}^n_+ \to \mathbb{R} : \Delta^k_a f \in L^p_a,\ k = 0, \dots, m \right\},$$

$$\|f\|_{W^{m,p}_{\Delta_a}} = \left(\sum_{k=0}^{m} \|\Delta^k_a f\|^p_{p,a} \right)^{\frac{1}{p}}.$$

Now we are in position to define our K-functional.

Definition 3. *Let $1 \le p \le \infty$. The K-functional of f at t with respect to the spaces L^p_a and $W^{k,p}_{\Delta_a}$ is*

$$K\left(f, t, L^p_a, W^{k,p}_{\Delta_a}\right) = \inf\{\|f - g\|_{p,a} + t\|\Delta^k_a g\|_{p,a} : g \in W^{k,p}_{\Delta_a}\},$$

where $f \in L^p_a$, $t > 0$.

The main theorem of this section is as follows.

Theorem 1. *Let $1 \le p \le \infty$. There is a positive constant $c = c(m, n, a, p)$, such that*

$$\frac{1}{c}\omega_{sph,m}(f, t)_{a,p} \le K\left(f, t^{2m}, L^p_a, W^{m,p}_{\Delta_a}\right) \le c\,\omega_{sph,m}(f, t)_{a,p},$$

where $f \in L^p_a$, $t > 0$.

This theorem is an n-dimensional version of [28, Theorem 1.2]. The main tool for the n-dimensional case is the spherical modulus of smoothness defined above. Usually, to prove results on equivalence of certain K-functionals and moduli of smoothness, or on the relation of moduli of smoothness and Besov- or potential space norms, modulus of smoothness is not averaged on the sphere (see e.g. [15]). Here, the symmetry of the space allows averaging the modulus of smoothness. Furthermore, averaging is essential to find the appropriate Hankel multiplier, which is crucial in the proof. Although the modulus of smoothness is spherical, the functions are not radial, thus the one-dimensional results are not applicable directly. However, by taking advantage of symmetry, we can follow the line of thought of the one-dimensional case. For sake of self-containedness we give the detailed proof.

For the proof of the inequality on the left-hand side, we need the following lemma.

Lemma 2. *If $g \in W^{k,p}_{\Delta_a}$, then for all $t > 0$ and $1 \le p \le \infty$*

$$\omega_{sph,m}(g,t)_{a,p} \le ct^{2m}\|\Delta_a^m g\|_{p,a},$$

where $c = c(a,n,m)$.

First, assuming Lemma 2, we prove the left-hand side of Theorem 1.

Proof. (of the left inequality of Theorem 1) Let $f \in L^p_a$ and $g \in W^{m,p}_{\Delta_a}$. According to (26) and (27) we have

$$\omega_{sph,m}(f,t)_{a,p} \le \omega_{sph,m}(f-g,t)_{a,p} + \omega_{sph,m}(g,t)_{a,p} \le 2^m\|f-g\|_{p,a} + ct^{2m}\|\Delta_a^m g\|_{p,a}$$
$$\le c\left(\|f-g\|_{p,a} + t^{2m}\|\Delta_a^m g\|_{p,a}\right),$$

where Lemma 2 is applied. Taking infimum over $g \in W^{m,p}_{\Delta_a}$, the left-hand side of Theorem 1 is proven. □

To prove Lemma 2 we need some definitions and further lemmas.

Notation.

First we define the radial function $\eta \in \mathcal{S}_e$ such that $|\eta| \le 1$,

$$\eta(x) = \eta(|x|) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| \ge 2 \end{cases}.$$

For an $f \in L^p_a$ $(1 \le p \le \infty)$, let

$$P_\nu(f)(x) := \mathcal{H}_a^{-1}\left(\eta\left(\frac{|\xi|}{\nu}\right)\hat{f}(\xi)\right) = \varrho_\nu *_a f(x), \tag{28}$$

where $\varrho_\nu(x) := \mathcal{H}_a^{-1}\left(\eta\left(\frac{|\xi|}{\nu}\right)\right)$.

Let us denote by E_e the set of even entire functions. We also define the space

$$M(\nu,p,a) := \{h \in L^p_a \cap E_e : \mathrm{supp}\hat{h} \subset B(0,\nu)\}.$$

Properties.

$$\|P_\nu(f)\|_{p,a} \le c\|f\|_{p,a}; \quad c \ne c(\nu). \tag{29}$$

Indeed, in view of (15) $\|P_\nu(f)\|_{p,a} \le \|\varrho_\nu\|_{1,a}\|f\|_{p,a}$. Since $\eta \in \mathcal{S}_e$, $\hat{\eta} \in \mathcal{S}_e \subset L^1_a$. Finally, by a simple replacement, it can be readily seen, that $\|\varrho_\nu\|_{1,a} = \|\varrho_1\|_{1,a}$.

$$\mathrm{supp}\hat{P}_\nu(f) \subset B(0,\nu). \tag{30}$$

Indeed, in distribution sense $\hat{P}_\nu(f) = \eta\left(\frac{|\xi|}{\nu}\right)\hat{f}(\xi)$, cf. [28, Lemma 4.5].

Since $\eta \in \mathcal{S}_e$, (29) and (30) ensure that $P_\nu(f) \in M(\nu, p, a)$.

$$P_\nu(h) = h, \quad \forall\ h \in M(\nu, p, a). \tag{31}$$

Indeed, if $\operatorname{supp}\hat{h} \subset B(0,\nu)$, $\hat{h}(\xi)\eta\left(\frac{|\xi|}{\nu}\right) = \hat{h}(\xi)$.

Moreover we define the best approximation of an $f \in L^p_a$ with functions from $M(\nu, p, a)$.

Definition 4. *Let $1 \le p \le \infty$.*

$$E_\nu(f)_{p,a} := \inf\{\|f-h\|_{p,a} : h \in M(\nu,p,a)\}.$$

A further important property of $P_\nu(f)$ is as follows.

$$\|f - P_\nu(f)\|_{p,a} \le cE_\nu(f)_{p,a}; \quad f \in L^p_a,\ c \ne c(f). \tag{32}$$

Indeed, let $h \in M(\nu,p,a)$ such that $\|f-h\|_{p,a} \le 2E_\nu(f)_{p,a}$. Then by (31) and (29) we have $\|f - P_\nu(f)\|_{p,a} \le \|f-h\|_{p,a} + \|P_\nu(f-h)\|_{p,a} \le cE_\nu(f)_{p,a}$.

For an element g of the Sobolev space, the norm of the iterated spherical difference of $P_\nu(g)$ can be estimated by the norm of the iterated Bessel-Laplacian of g.

Lemma 3. *Let $1 \le p \le \infty$ and $g \in W^{m,p}_{\Delta_a}$. Then we have*

$$\|\Delta^m_{sph,h,a}P_\nu(g)\|_{p,a} \le ch^{2m}\|\Delta^m_a g\|_{p,a},$$

where $c = c(n,a,m)$.

Proof. According to (25) we have

$$\mathcal{H}_a\left(\Delta^m_{sph,h,a}P_\nu(g)\right)(\xi) = \left(1 - j_{\frac{n+|a|}{2}-1}(h|\xi|)\right)^m \eta\left(\frac{|\xi|}{\nu}\right)\hat{g}(\xi)$$

$$= h^{2m}\eta\left(\frac{|\xi|}{\nu}\right)\frac{\left(1 - j_{\frac{n+|a|}{2}-1}(h|\xi|)\right)^m}{(-1)^m(h|\xi|)^{2m}}(-1)^m|\xi|^{2m}\hat{g}(\xi).$$

According to (24) $u_h := \mathcal{H}_a^{-1}\left(\frac{\left(1-j_{\frac{n+|a|}{2}-1}(h|\xi|)\right)}{(h|\xi|)^2}\right) \in L^1_a$ and its norm is independent of h, so by (15) the same is valid for $u_h *_a \cdots *_a u_h =: \psi_h$. That is by (15) and (18) we have

$$\|\Delta^m_{sph,h,a}P_\nu(g)\|_{p,a} \le h^{2m}\|\varrho_\nu\|_{1,a}\|\psi_h\|_{1,a}\|\Delta^m_a g\|_{p,a} \le ch^{2m}\|\Delta^m_a g\|_{p,a},$$

where $c = \|\varrho_1\|_{1,a}\|\psi_1\|_{1,a}$. □

We also need a similar estimation on the norm of the modified iterated spherical difference of g.

Lemma 4. *Let $1 \le p \le \infty$ and $g \in W^{m,p}_{\Delta_a}$. Then we have*

$$\|\tilde{\Delta}^m_{sph,h,a}g\|_{p,a} \le ch^{2m}\|\Delta^m_a g\|_{p,a},$$

where $c = c(n,a,m)$.

Proof. By Fubini's theorem, (17) and (19) we have

$$\mathcal{H}_a\left(\tilde{\Delta}^m_{sph,h,a}g\right)(\xi)=\hat{g}(\xi)\mathcal{J}_{\frac{n+|a|}{2}-1,m}(h|\xi|).$$

In view of (24) $\mathcal{H}_a^{-1}\left(\frac{\mathcal{J}_{\frac{n+|a|}{2}-1,m}(h|\xi|)}{|\xi|^{2m}}\right)\in L^1_a$, and $\left\|\mathcal{H}_a^{-1}\left(\frac{\mathcal{J}_{\frac{n+|a|}{2}-1,m}(h|\xi|)}{|\xi|^{2m}}\right)\right\|_{1,a}=$ $h^{2m}\left\|\mathcal{H}_a^{-1}\left(\frac{\mathcal{J}_{\frac{n+|a|}{2}-1,m}(|\xi|)}{|\xi|^{2m}}\right)\right\|_{1,a}$. Thus by (15) we have

$$\|\tilde{\Delta}^m_{sph,h,a}g\|_{p,a}=\left\|\mathcal{H}_a^{-1}\left(\frac{\mathcal{J}_{\frac{n+|a|}{2}-1,m}(h|\xi|)}{|\xi|^{2m}}\right)*_a\Delta^m_a g\right\|_{p,a}\le ch^{2m}\|\Delta_a^m g\|_{p,a},$$

where $c=\left\|\mathcal{H}_a^{-1}\left(\frac{\mathcal{J}_{\frac{n+|a|}{2}-1,m}(|\xi|)}{|\xi|^{2m}}\right)\right\|_{1,a}$. □

Lemma 5. *Let $1\le p\le\infty$ and $g\in W^{m,p}_{\Delta_a}$. Then we have*

$$E_\nu(g)_{p,a}\le\frac{c}{\nu^{2m}}\|\Delta_a^m g\|_{p,a},$$

where $c=c(n,a,m)$.

Proof. In view of (23) for the modified iterated difference we have

$$-\tilde{\Delta}^m_{y,a}g(x):=\sum_{l=1}^m(-1)^{l-1}\binom{m}{l}T_a^{\sqrt{l}y}g(x)-g(x)=\sum_{l=1}^m d_lT_a^{\sqrt{l}y}g(x)-g(x),$$

where $\sum_{l=1}^m d_l=1$.

Let $\tilde{\eta}\in M(1,1,a)$ be a nonnegative radial function such that $\|\tilde{\eta}\|_{1,a}=1$, moreover $\int_0^\infty\tilde{\eta}(\varrho)\varrho^{2m+|a|+n-1}d\varrho<\infty$. (E.g. $\tilde{\eta}(x)=C\left(\frac{\sin|x|^2}{|x|^2}\right)^M$, where $M\in\mathbb{N}$, $|a|+n+2m<2M$.) Let

$$Q_{\nu,m}(g)(x):=\int_{\mathbb{R}^n_+}\tilde{\eta}(y)\left(-\tilde{\Delta}^m_{\frac{y}{\nu},a}g(x)+g(x)\right)y^ady=\sum_{l=1}^m d_l\int_{\mathbb{R}^n_+}T_a^{\sqrt{l}\frac{y}{\nu}}g(x)\tilde{\eta}(y)y^ady$$

$$=\sum_{l=1}^m d_l\int_{\mathbb{R}^n_+}T^x_ag\left(\sqrt{l}\frac{y}{\nu}\right)\tilde{\eta}(y)y^ady=\int_{\mathbb{R}^n_+}\left(\sum_{l=1}^m d_l\left(\frac{\nu}{\sqrt{l}}\right)^{n+|a|}\tilde{\eta}\left(\frac{\nu}{\sqrt{l}}t\right)\right)T^x_ag(t)t^adt$$

$$=:K_{\nu,m,a}*_a g(x).$$

Now we estimate the difference of g and $Q_\nu(g)$. By Minkowski's inequality we have

$$\|g-Q_{\nu,m}(g)\|_{p,a}=\left(\int_{\mathbb{R}^n_+}\left|\int_0^\infty\tilde{\eta}(\varrho)\int_{S^n_{1+}}\tilde{\Delta}^m_{\frac{\varrho\Theta}{\nu},a}g(x)\Theta^adS(\Theta)\varrho^{|a|+n-1}d\varrho\right|^px^adx\right)^{\frac{1}{p}}$$

$$\le|S^n_{1+}|_a\int_0^\infty\tilde{\eta}(\varrho)\varrho^{|a|+n-1}\|\tilde{\Delta}^m_{sph,\frac{\varrho\Theta}{\nu},a}g(x)\|_{p,a}d\varrho.$$

Thus by the assumption on $\tilde{\eta}$ and by Lemma 4 we have

$$\|g-Q_{\nu,m}(g)\|_{p,a}\le\frac{c}{\nu^{2m}}\int_0^\infty\tilde{\eta}(\varrho)\varrho^{|a|+n-1+2m}d\varrho\|\Delta_a^m g\|_{p,a}\le\frac{c}{\nu^{2m}}\|\Delta^m_a g\|_{p,a}.$$

Finally we have to show that $Q_{\nu,m}(g) \in M(\nu, p, a)$. $K_{\nu,m,a} \in E_e$ and so $Q_{\nu,m}(g) \in E_e$.

$$\|Q_{\nu,m}(g)\|_{p,a} \le \|K_{\nu,m,a}\|_{1,a}\|g\|_{p,a}.$$

Since

$$\|K_{\nu,m,a}\|_{1,a} \le \sum_{l=1}^{m} |d_l| \left(|S^n_{1+}|_a \int_0^\infty \left(\frac{\nu}{\sqrt{l}}\right)^{n+|a|} \tilde{\eta}\left(\frac{\nu}{\sqrt{l}} r\right) r^{n+|a|-1} dr \right)$$
$$= c(n, a, m)\|\tilde{\eta}\|_{1,a},$$

where

$$c(n, a, m) < 2^m |S^n_{1+}|_a.$$

Since $\tilde{\eta} \in M(1, 1, a)$, $\operatorname{supp}\hat{K}_{\nu,m,a} \subset B(0, \nu)$. Thus $\operatorname{supp}\hat{Q}_{\nu,m}(g) \subset B(0, \nu)$, which finishes the proof. □

Proof. (of Lemma 2) In view of (12)

$$\|\Delta^m_{sph,h,a} g\|_{p,a} \le \|\Delta^m_{sph,h,a}(g - P_\nu(g)\|_{p,a} + \|\Delta^m_{sph,h,a} P_\nu(g)\|_{p,a}$$
$$\le 2^m \|g - P_\nu(g)\|_{p,a} + \|\Delta^m_{sph,h,a} P_\nu(g)\|_{p,a}.$$

According to (32), Lemma 5 and Lemma 3 if $h \in (0, t]$, $\nu = \frac{2}{h}$, we have

$$\|\Delta^m_{sph,h,a} g\|_{p,a} \le 2^{2m} \frac{c}{\nu^{2m}} \|\Delta^m_a g\|_{p,a} + h^{2m} \|\Delta^m_a g\|_{p,a} \le c t^{2m} \|\Delta^m_a g\|_{p,a},$$

where $c \ne c(m)$. The proof is finished by taking supremum. □

To prove the right-hand side inequality in Theorem 1 we need the following lemmas.

Lemma 6. *Let $f \in L^p_a$ $1 \le p \le \infty$, $\nu > 0$, $m \in \mathbb{N}$. Then we have*

$$\|\Delta^m_a P_\nu(f)\|_{p,a} \le c(a, m, n)\nu^{2m} \omega_{sph,m}\left(f, \frac{1}{\nu}\right)_{p,a}.$$

Lemma 7. *With the notation of the previous lemma we have*

$$\|f - P_{\frac{\nu}{2}}(f)\|_{p,a} \le c(a, m, n)\|\Delta^m_{sph,\frac{1}{\nu},a} f\|_{p,a}.$$

Proof. (of the right-hand side of Theorem 1) Since $P_\nu(f) \in M(\nu, p, a)$ (cf. (29) and (30)), by definition we have

$$K\left(f, t^{2m}, L^p_a, W^{m,p}_{\Delta_a}\right) \le \|f - P_\nu(f)\|_{p,a} + t^{2m}\|\Delta^m_a P_\nu(f)\|_{p,a}.$$

In view of (32) and (28) we have

$$\|f - P_\nu(f)\|_{p,a} \le c E_\nu(f)_{p,a} \le c\|f - P_{\frac{\nu}{2}}(f)\|_{p,a}.$$

Let $\nu = \frac{1}{t}$. Taking into consideration that $\|\Delta^m_{sph,\frac{1}{\nu},a} f\|_{p,a} \le \omega_{sph,m}\left(f, \frac{1}{\nu}\right)_{p,a}$, Lemma 6 and Lemma 7 imply

$$\|f - P_\nu(f)\|_{p,a} + t^{2m}\|\Delta^m_a P_\nu(f)\|_{p,a}$$
$$\le c\omega_{sph,m}\left(f, \frac{1}{\nu}\right)_{p,a} + c t^{2m}\nu^{2m}\omega_{sph,m}\left(f, \frac{1}{\nu}\right)_{p,a} = c\omega_{sph,m}\left(f, \frac{1}{\nu}\right)_{p,a}.$$

□

Proof. (of Lemma 6) It is enough to prove that for all $h \in \left(0, \frac{1}{2\nu}\right]$,

$$\|\Delta_a^m P_\nu(f)\|_{p,a} \le c\frac{1}{h^{2m}}\|\Delta_{sph,h,a}^m f\|_{p,a}. \tag{33}$$

Indeed, if (33) is fulfilled, taking $h = \frac{1}{2\nu}$, then taking supremum and considering the monotonicity of the modulus of smoothness, the lemma is verified.
To prove (33) let take the Hankel transform of the left-hand side. Since $h \le \frac{1}{2\nu}$, $\eta\left(\frac{|\xi|}{\nu}\right) = \eta\left(\frac{|\xi|}{\nu}\right)\eta(h|\xi|)$, thus we have

$$\mathcal{H}_a(\Delta_a^m P_\nu(f)) = \frac{1}{h^{2m}}\eta\left(\frac{|\xi|}{\nu}\right)\frac{(-1)^m(h|\xi|)^{2m}\eta(h|\xi|)}{(1-j_{\frac{n+|a|}{2}-1}(h|\xi|))^m}(1-j_{\frac{n+|a|}{2}-1}(h|\xi|))^m\hat{f}(\xi).$$

Recall that

$$\mathcal{H}_a(\Delta_{sph,h,a}^m f) = (1-j_{\frac{n+|a|}{2}-1}(h|\xi|))^m\hat{f}(\xi).$$

Let us take into consideration that

$$\frac{(-1)^m(h|\xi|)^{2m}\eta(h|\xi|)}{(1-j_{\frac{n+|a|}{2}-1}(h|\xi|))^m} \in \mathcal{S}_e,$$

and

$$\left\|\mathcal{H}_a^{-1}\left(\frac{(-1)^m(h|\xi|)^{2m}\eta(h|\xi|)}{(1-j_{\frac{n+|a|}{2}-1}(h|\xi|))^m}\right)\right\|_{1,a} = \left\|\mathcal{H}_a^{-1}\left(\frac{(-1)^m(|\xi|)^{2m}\eta(|\xi|)}{(1-j_{\frac{n+|a|}{2}-1}(|\xi|))^m}\right)\right\|_{1,a}.$$

Since

$$\Delta_a^m P_\nu(f) = \frac{1}{h^{2m}}\varrho_\nu *_a \mathcal{H}_a^{-1}\left(\frac{(-1)^m(h|\xi|)^{2m}\eta(h|\xi|)}{(1-j_{\frac{n+|a|}{2}-1}(h|\xi|))^m}\right) *_a \Delta_{sph,h,a}^m f,$$

by (15) we obtain (33). □

Proof. (of Lemma 7) We start again with Hankel transform.

$$\mathcal{H}_a\left(f - P_{\frac{\nu}{2}}(f)\right) = \frac{1-\eta\left(\frac{2|\xi|}{\nu}\right)}{\left(1-j_{\frac{n+|a|}{2}-1}\left(\frac{|\xi|}{\nu}\right)\right)^m}\left(1-j_{\frac{n+|a|}{2}-1}\left(\frac{|\xi|}{\nu}\right)\right)^m\hat{f}(\xi)$$

$$= \varphi\left(\frac{|\xi|}{\nu}\right)\mathcal{H}_a\left(\Delta_{sph,h,a}^m f\right).$$

φ is decomposed by $\varphi_1 + \varphi_2$, cf. [28, (4.67)], where

$$\varphi_1 = (1-\eta(2|\xi|))\left((1-j_{\frac{n+|a|}{2}-1}(|\xi|))^{-m} - Q_N(j_{\frac{n+|a|}{2}-1}(|\xi|))\right),$$

and

$$\varphi_2 = (1-\eta(2|\xi|))Q_N(j_{\frac{n+|a|}{2}-1}(|\xi|)).$$

Here $Q_N(u)$ is a polynomial of degree N, and is defined as follows. With $N = N(n,a)$ large enough, $\frac{1}{(1-u)^m} = Q_N(u) + \sum_{j=N+1}^\infty b_j u^j$. In [28, (4.68)-(4.70)] is proved that

$$\left\|\mathcal{H}_{\frac{n+|a|}{2}-1}^{-1}\left(\varphi_1\left(\frac{r}{\nu}\right)\right)\right\|_{1,\frac{n+|a|}{2}-1,\mathbb{R}_+} = \left\|\mathcal{H}_{\frac{n+|a|}{2}-1}^{-1}\left(\varphi_1(r)\right)\right\|_{1,\frac{n+|a|}{2}-1,\mathbb{R}_+}$$

$$= \left\| \mathcal{H}_a^{-1} \left(\varphi_1 \left(\frac{|\xi|}{\nu} \right) \right) \right\|_{1,a,\mathbb{R}^n_+} = \left\| \mathcal{H}_a^{-1} \left(\varphi_1 \left(|\xi| \right) \right) \right\|_{1,a,\mathbb{R}^n_+} < \infty, \tag{34}$$

i.e. the norm is finite and independent of ν.
To treat φ_2 we introduce the spherical translation as follows.

$$T^r_{sph,a} f(x) := \frac{1}{|S^n_{1+}|_a} \int_{S^n_{1+}} T^{r\Theta} f(x) \Theta^a dS(\Theta). \tag{35}$$

It is clear that

$$\mathcal{H}_a((T^r_{sph,a})^k f)(\xi) = j^k_{\frac{n+|a|}{2}-1}(r|\xi|) \hat{f}(\xi).$$

Moreover in view of (12) by Minkowski's inequality we have

$$\|T^r_{sph,a} f(x)\|_{p,a} \leq \frac{1}{|S^n_{1+}|_a} \int_{S^n_{1+}} \|T^{r\Theta} f(x)\|_{p,a} \Theta^a dS(\Theta) \leq \|f\|_{p,a}. \tag{36}$$

Thus for any $g \in \mathcal{S}'_e$

$$\varphi_2 \left(\frac{|\xi|}{\nu} \right) \hat{g}(\xi) = \sum_{k=0}^N b_k j^k_{\frac{n+|a|}{2}-1} \left(\frac{|\xi|}{\nu} \right) \hat{g}(\xi) + \sum_{k=0}^N b_k \eta \left(\frac{2|\xi|}{\nu} \right) j^k_{\frac{n+|a|}{2}-1} \left(\frac{|\xi|}{\nu} \right) \hat{g}(\xi)$$

$$= \mathcal{H}_a \left(\sum_{k=0}^N b_k (T^r_{sph,a})^k g + \varrho_{\frac{\nu}{2}} *_a \sum_{k=0}^N b_k (T^r_{sph,a})^k g \right).$$

Finally we have

$$\|f - P_{\frac{\nu}{2}}(f)\|_{p,a} \leq \left\| \mathcal{H}_a^{-1} \left(\varphi_1 \left(\frac{|\xi|}{\nu} \right) \right) *_a \Delta^m_{sph,h,a} f \right\|_{p,a}$$

$$+ \left\| \sum_{k=0}^N b_k (T^r_{sph,a})^k \Delta^m_{sph,h,a} f + \varrho_{\frac{\nu}{2}} *_a \sum_{k=0}^N b_k (T^r_{sph,a})^k \Delta^m_{sph,h,a} f \right\|_{p,a}.$$

Thus by (29), (34), (36) and (15) we have

$$\|f - P_{\frac{\nu}{2}}(f)\|_{p,a}$$

$$\leq \left(\|\mathcal{H}_a^{-1}(\varphi_1(|\xi|))\|_{1,a} + (1 + \|\varrho_1\|_{1,a}) \sum_{k=0}^N b_k \right) \|\Delta^m_{sph,h,a} f\|_{p,a} = c \|\Delta^m_{sph,h,a} f\|_{p,a}.$$

□

We mention that Lemma 7 implies the next estimation. Although we will not use it further, it is interesting in itself from an approximation theory perspective.

Corollary 1. *Let $f \in L^p_a$ $1 \leq p \leq \infty$, $\nu > 0$, $m \in \mathbb{N}$. Then we have*

$$E_\nu(f)_{p,a} \leq c \omega_{sph,m} \left(f, \frac{1}{\nu} \right)_{p,a}.$$

4. Bessel-Sobolev- and fractional Bessel-Sobolev spaces

Classical Sobolev spaces contain functions that, together with some of their derivatives, lie in a certain Banach space. Definition 2 is a natural extension of this definition, i.e. instead of derivatives we use the Bessel differential operator. As shown in the previous section (see also [28]), from an approximation theoretic perspective it is worth introducing the Sobolev spaces generated by the Bessel operator. The remark below shows that these spaces are definitely different from the classical ones. Applying the results of the previous section, here we define fractional Bessel-Sobolev spaces. The necessity of these spaces arose during the study of fractional Bessel-type operators. The last section contains these investigations.

First, recalling Definition 2 we make the following observations.

Remark.

(1) Take the next weighted Sobolev space:
$W_a^{m,p}(\mathbb{R}_+^n) := \left\{f : \mathbb{R}_+^n \to \mathbb{R} : D^\alpha f \in L_a^p, \ |\alpha| \leq m\right\}$. Notice, that the Bessel-Sobolev spaces are not coincide with the weighted Sobolev spaces. For instance, consider the fundamental solution of the Laplace-Bessel operator, i.e. the solution to $\Delta_a E = \delta_a$, where δ_a stands for the weighted Dirac delta distribution, for which $\langle \delta_a, \varphi\rangle_a = \varphi(0)$. Then according to [32, Theorem 93]

$$E(x) = c(n,a)\left\{\begin{array}{l} \ln|x|, \ \ n+|a| = 2 \\ |x|^{2-n-|a|}, \ \ n+|a|>2. \end{array}\right.$$

Let

$$f(x) := \left\{\begin{array}{l} E(x), \ \ |x|<1 \\ 0, \ \ |x|>2, \end{array}\right.$$

and let f be radial, $f \in C^2(\mathbb{R}_+^n)$. Then $f, \Delta_a f \in L_a^p$ if $1 \leq p < 1 + \frac{2}{n+|a|-2}$, $\frac{\partial}{\partial x_i} f \in L_a^p$ if $1 \leq p < 1 + \frac{1}{n+|a|-1}$ and $\frac{\partial^2}{\partial x_i \partial x_j} f \in L_a^p$ if $1 \leq p < 1 + \frac{1}{n+|a|}$. That is $f \in W_{\Delta_a}^{1,p}(\mathbb{R}_+^n)$, but $f \notin W_a^{2,p}(\mathbb{R}_+^n)$ if $1 + \frac{1}{n+|a|} \leq p < 1 + \frac{2}{n+|a|-2}$.
On the other hand let $g(x) = e^{-\sum_{i=1}^n x_i}$. Then $g \in W_a^{2,p}(\mathbb{R}_+^n)$ for all $1 \leq p \leq \infty$, but $g \notin W_{\Delta_a}^{1,p}(\mathbb{R}_+^n)$ if $p \geq n + |a|$.

(2) Similarly to the classical case $W_{\Delta_a}^{m,p}((R_+^n)$ are Banach spaces (cf.[29, Proposition 3.2]) which are separable if $1 \leq p < \infty$ and are reflexive and uniformly convex if $1 < p < \infty$, cf. [1, 3.4].

By interpolation of spaces (K-method, see [7, (3), page 39]), L_a^p and $W_{\Delta_a}^{1,p}$ we arrive to the so-called Besov-type seminorm. If $p = q$, it is

$$\left(\int_0^\infty \left(\frac{K\left(f, u, L_a^p, W_{\Delta_a}^{1,p}\right)}{u^\Theta}\right)^p \frac{du}{u}\right)^{\frac{1}{p}}.$$

Replacing $u = t^2$ we get the following definition.
Let $0 < s < 2$.

$$\int_0^\infty \left(\frac{K\left(f, t^2, L_a^p, W_{\Delta_a}^{1,p}\right)}{t^s}\right)^p \frac{dt}{t} \sim \int_0^\infty \left(\frac{\omega_{sph,1}(f,t)_{a,p}}{t^s}\right)^p \frac{dt}{t}$$

$$= \int_0^\infty \left(\frac{\sup_{0<h<t} \|\Delta_{sph,h,a} f\|_{p,a}}{t^s}\right)^p \frac{dt}{t} =: \|f\|^p_{B^s_{\Delta_a,1,p,p}}. \tag{37}$$

We also define the spherical Gagliardo seminorm:

$$[f]_{sph,s,p,a} := \left(\int_0^\infty \left(\frac{\|\Delta_{sph,t,a} f\|_{p,a}}{t^s}\right)^p \frac{dt}{t}\right)^{\frac{1}{p}} \tag{38}$$

$$= \frac{1}{|S^n_{1+}|_a}\left(\int_0^\infty \frac{\int_{\mathbb{R}^n_+}\left|\int_{S^n_{1+}}(f(x)-T^{t\Theta}_a f(x))\Theta^a dS(\Theta)\right|^p x^a dx}{t^{sp+1}} dt\right)^{\frac{1}{p}}$$

$$= \frac{1}{|S^n_{1+}|_a}\left(\int_{\mathbb{R}^n_+}\int_0^\infty \frac{\left|\int_{S^n_{1+}}(f(x)-T^{t\Theta}_a f(x))\Theta^a dS(\Theta)\right|^p}{t^{sp+n+|a|}} t^{n+|a|-1} dt x^a dx\right)^{\frac{1}{p}}$$

$$\leq \frac{1}{|S^n_{1+}|_a}\left(\int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \frac{|f(x)-T^y_a f(x)|^p}{|y|^{sp+n+|a|}} y^a dy x^a dx\right)^{\frac{1}{p}} =: [f]_{s,p,a}, \tag{39}$$

which is the weighted Gagliardo seminorm. As a corollary of Theorem 1, the Besov-type seminorm is equivalent with with the spherical Gagliardo seminorm.

Corollary 2. *If $f \in L^p_a$, $s \in (0,2)$,*

$$\|f\|_{B^s_{\Delta_a,1,p,p}} \sim [f]_{sph,s,p,a}.$$

Proof. Of course, by Definition 1, $[f]_{s,\gamma,p,a} \leq \|f\|_{B^s_{\Delta_a,1,p,p}}$. On the other hand in view of Definition 3 we have

$$\|f\|^p_{B^s_{\Delta_a,1,p,p}} \sim \int_0^\infty \frac{\left(\inf\{\|f-g\|_{p,a} + t^2\|\Delta_a g\|_{p,a} : g \in W^{1,p}_{\Delta_a}\}\right)^p}{t^{sp+1}} dt.$$

In view of (28)

$$\|f\|_{B^s_{\Delta_a,1,p,p}} \leq c\int_0^\infty \frac{\left(\|f-P_{\frac{1}{2t}}(f)\|_{p,a} + t^2\|\Delta_a P_{\frac{1}{2t}}(f)\|_{p,a}\right)^p}{t^{sp+1}} dt.$$

According to Lemma 7 and (33) we have

$$\|f\|^p_{B^s_{\Delta_a,1,p,p}} \leq c\int_0^\infty \frac{\left(\|\Delta_{sph,t,a} f\|_{p,a} + \frac{t^2}{t^2}\|\Delta_{sph,t,a} f\|_{p,a}\right)^p}{t^{sp+1}} dt$$

$$= c(n,p,a)[f]^p_{sph,s,p,a}.$$

□

Thus we define the fractional Bessel-Sobolev spaces as follows.

Definition 5. *Let $\frac{s}{2} \in (0,1)$, $1 \leq p < \infty$. The fractional Bessel-Sobolev space is*

$$W^{\frac{s}{2},p}_{\Delta_a} := \left\{f \in L^p_a(\mathbb{R}^n_+) : \|f\|_{W^{\frac{s}{2},p}_{\Delta_a}} < \infty\right\}, \tag{40}$$

where

$$\|f\|_{W^{\frac{s}{2},p}_{\Delta_a}} = \left(\|f\|^p_{p,a} + [f]^p_{sph,s,p,a}\right)^{\frac{1}{p}}. \tag{41}$$

Remark.

The so-called K-method ensures that $W^{\frac{s}{2},p}_{\Delta_a}$ is also Banach space, cf. e.g. [7, Theorem 3.4.2].

The theorem below will be useful during the following investigations.

Theorem 2. *Let* $0<\frac{s}{2}<1$*;* $1\le p<\infty$*. Then* $\mathcal{S}_e$ *is a dense subset of* $W^{1,p}_{\Delta_a}(\mathbb{R}^n_+)$ *and* $W^{\frac{s}{2},p}_{\Delta_a}(\mathbb{R}^n_+)$*.*

Proof. Obviously $\mathcal{S}_e\subset W^{1,p}_{\Delta_a}$, and independently, Lemma 2 ensures that $\mathcal{S}_e\subset W^{\frac{s}{2},p}_{\Delta_a}$.

First we prove that $C_0^\infty(\mathbb{R}^n_+)$ is dense in $W^{1,p}_{\Delta_a}(\mathbb{R}^n_+)$. Following the standard chain of ideas we use Bessel convolution instead of the standard one. Let $\tilde\varepsilon>0$ be arbitrary, $u\in W^{1,p}_{\Delta_a}(\mathbb{R}^n_+)$. Then there are δ and R (depending on u and $\tilde\varepsilon$) such that $\|u\|_{W^{1,p}_{\Delta_a},\mathbb{R}^n_+\setminus\Omega_{\tilde\varepsilon}}<\tilde\varepsilon$, where $\Omega_{\tilde\varepsilon}=\mathbb{R}^n_+\cap B(0,R)\setminus\{x\in\mathbb{R}^n_+ : \operatorname{dist}(x,\partial\mathbb{R}^n_+)\le\delta\}$. Let us cover $\overline{\Omega_{\tilde\varepsilon}}$ with balls of radius at most $\frac{\delta}{2}$. Select a finite covering $\Omega_{\tilde\varepsilon}\subset\cup_{j=1}^m B_j$ and let $\{\Psi_j\}_{j=1}^m$ be the corresponding partition of unity. Define $u_j:=\Psi_j u$, and $\eta_{a,\varepsilon}(x):=\frac{k_a}{\varepsilon^{n+|a|}}\eta\left(\frac{x}{\varepsilon}\right)$, where $\eta(x)=k_ae^{-\frac{1}{1-|x|^2}}$, if $|x|<1$, $\eta(x)=0$ if $|x|\ge1$ and $\int_{\mathbb{R}^n_+}\eta(x)x^a=1$. Let $\varepsilon<\frac{\delta}{2}$ and for sake of abbreviation let $d(x,t,\vartheta)=\sum_{i=1}^n x_i^2+t_i^2-2x_it_i\cos\vartheta_i$. Considering that if $\varepsilon<\|x-t\|\le\sqrt{d(x,t,\vartheta)}$, then $T^t\eta_{a,\varepsilon}(x)=0$, $\operatorname{supp}\eta_{a,\varepsilon}(x)*_a u_j\subset\mathbb{R}^n_+$, so $\eta_{a,\varepsilon}(x)*_a u_j\in C_0^\infty(\mathbb{R}^n_+)$.
Let $\varphi_j\in C_0^\infty(\mathbb{R}^n_+)$ such that $\|u_j-\varphi_j\|_{p,a}\le\epsilon_j$. Then

$$\|u_j-\eta_{a,\varepsilon}(x)*_a u_j\|_{p,a}\le\|u_j-\varphi_j\|_{p,a}\left(1+\|\eta_{a,\varepsilon}\|_{1,a}\right)+\|\varphi_j-\eta_{a,\varepsilon}*_a\varphi_j\|_{p,a}.$$

Since $\|\eta_{a,\varepsilon}\|_{1,a}=\|\eta_{a,1}\|_{1,a}$, i.e. is independent of ε, it is enough to deal with the second term above. Taking into account the norm of the Bessel translation we have

$$|\varphi_j-\eta_{a,\varepsilon}*_a\varphi_j(x)|=\left|\int_{\mathbb{R}^n_+}(\varphi_j(x)-\varphi_j(t))T^t\eta_{a,\varepsilon}(x)t^adt\right|\le\sup_{|x-t|<\varepsilon}|\varphi(x)-\varphi(t)|,$$

thus

$$\|\varphi_j-\eta_{a,\varepsilon}*_a\varphi_j\|_{p,a}\le2\epsilon_j|\operatorname{supp}\varphi_j|_a.$$

In view of (50) we have $\Delta_a(\eta_{a,\varepsilon}(x)*_a u_j)=\Delta_a\eta_{a,\varepsilon}(x)*_a u_j=\eta_{a,\varepsilon}(x)*_a\Delta_au_j$, in distribution sense. Since $u\in W^{1,p}_{\Delta_a}$, $\Delta_au_j\in L^p_a$. Indeed, recalling that for all $\varphi\in\mathcal{S}_e$, $u\in\mathcal{S}_e'$ $\langle\Delta_au,\varphi\rangle_a=\langle u,\Delta_a\varphi\rangle_a$, we have
$\|\Delta_au\|_{p,a}=\sup_{\varphi\in\mathcal{S}_e,\|\varphi\|_{p',a}\le1}\int_{\mathbb{R}^n_+}|u(x)\Delta_a\varphi(x)|x^adx=:K$. Thus we get

$$\left|\int_{\mathbb{R}^n_+}\Delta_au_j\varphi x^adx\right|=\left|\int_{\mathbb{R}^n_+}u_j\Delta_a\varphi x^adx\right|\le\|\psi_j\|_\infty K.$$

That is we can apply the previous part to Δ_au_j. Since the partition is finite, choosing $\tilde\varepsilon$, ε, ϵ small enough, $\sum_{j=1}^m\eta_{a,\varepsilon}(x)*_a u_j$ is can be arbitrarily close to u in $W^{1,p}_{\Delta_a}(\mathbb{R}^n_+)$-norm.

Let $0<s<2$. According to [1, Lemma 7.18] it is enough to define an appropriate sequence of linear operators. Let $D_k=\mathbb{R}^n_+\cap B(0,k)\setminus\{x\in\mathbb{R}^n_+ : \operatorname{dist}(x,\partial\mathbb{R}^n_+)\le\frac{1}{k}\}$. With the above procedure we construct $\varphi_k:=\sum_{j=1}^{m_k}\eta_{a,\varepsilon_k}*_au_j\in C_0^\infty(\mathbb{R}^n_+)$, such that $\|u-\varphi_k\|_{W^{1,p}_{\Delta_a},D_k}<\frac{1}{k}$. Since $\|u\|_{W^{1,p}_{\Delta_a},\mathbb{R}^n_+\setminus D_k}\to0$ when $k\to\infty$, with $P_ku:=\varphi_k$ we have that $\|P_ku-u\|_{B_i}\to0$, where $B_1=L^p_a(\mathbb{R}^n_+)$, $B_2=W^{1,p}_{\Delta_a}(\mathbb{R}^n_+)$, so $\mathcal{S}_e\subset B_1\cap B_2$ is dense in $W^{\frac{s}{2},p}_{\Delta_a}(\mathbb{R}^n_+)$, $0<\frac{s}{2}<1$. □

5. Bessel B-potential spaces

Bessel potential spaces were introduced in the 1960s, see [4]. These intermediate spaces of the Sobolev ones is proven to be an appropriate tool of studying partial differential equations. In connection with the fractional Laplace operator, Riesz potential spaces were first introduced as the natural object of classical potential theory. Indeed, denoting by $I_s g := c|x|^{s-n} * g$, for a function from the Schwartz class, $f \in \mathcal{S}$ we have $f = I_s g = \mathcal{F}^{-1}(\frac{1}{|\xi|^s}\hat{g}(\xi)) = \mathcal{F}^{-1}(\frac{1}{|\xi|^s}|\xi|^s\hat{f}(\xi)) = I_s(-\Delta)^{\frac{s}{2}} f$. Unfortunately, the operator I_s is not bounded on L^p. It turned out that this problem can be avoided by taking the fractional operator $(I-\Delta)^{\frac{s}{2}}$ together with the Bessel potential, i.e. if $f \in \mathcal{S}$, $f = \mathcal{G}_s g = \mathcal{F}^{-1}(\frac{1}{(1+|\xi|^2)^{\frac{s}{2}}}\hat{g}(\xi))$, where $g = (I-\Delta)^{\frac{s}{2}} f$ and $\mathcal{G}_s g = G_s * g$, where $G_s = \mathcal{F}^{-1}(\frac{1}{(1+|\xi|^2)^{\frac{s}{2}}})$ is the Bessel kernel (cf. (42) with $a = 0$), and so $\mathcal{G}_s$ is bounded in L^p for any $1 \le p \le \infty$.

Thus, according to the original definition of Calderón, $L^{s,p}$ spaces consist of all functions $\varphi = G_s * f$, where $f \in L^p$. This radially symmetric world was consistent with the Laplace operator, as well as classical and p-potential theory. The inclusion of another partial differential operator causes changes in the symmetry of the space. This justifies the modification of the translation, and in parallel, the modification of the convolution structure and the associated integral transformation. Since the Bessel translation and the Bessel operator are interchangeable, examination of Bessel-type differential operators leads to the notion of Bessel B-potential spaces.

To introduce Bessel B-potential spaces (actually, Bessel Bessel-potential spaces), we define the appropriate kernel function $G_{a,\nu}$. The Bessel kernel is

$$G_{a,\nu}(x) := \frac{2^{\frac{n-|a|-\nu}{2}+1}}{\Gamma\left(\frac{\nu}{2}\right)\prod_{i=1}^{n}\Gamma(\alpha_i+1)} \frac{K_{\frac{n+|a|-\nu}{2}}(|x|)}{|x|^{\frac{n+|a|-\nu}{2}}}, \tag{42}$$

where K_ν is defined by (6).

(7) and (8) imply that

$$G_{a,\nu} \in L^1_a(\mathbb{R}^n_+), \quad \text{if } \nu > 0. \tag{43}$$

Moreover, in accordance with the relation of Fourier and Hankel transform of a radial function we have

$$G_{a,\nu}(x) = \mathcal{H}_a^{-1}\left(\frac{1}{(1+|\xi|^2)^{\frac{\nu}{2}}}\right), \tag{44}$$

see e.g. [13, (5)].

Below we also need the fractional partial differential operators $(-\Delta_a)^{\frac{s}{2}}$ and $(I-\Delta_a)^{\frac{s}{2}}$, where $0 < s < 2$. Let $\varphi \in \mathcal{S}_e$.

$$(-\Delta_a)^{\frac{s}{2}}\varphi(x) = C(n,a,s)\int_{\mathbb{R}^n_+} \frac{\varphi(x) - T_a^y\varphi(x)}{|y|^{n+|a|+s}} y^a dy, \tag{45}$$

see [23] and in one dimension [8].

In view of [13, (46)] we have

$$(I-\Delta_a)^{\frac{s}{2}}\varphi(x) = \int_{\mathbb{R}^n_+} (T_a^y\varphi(x) - \varphi(x))G_{a,-s}(|y|)y^a dy + \varphi(x). \tag{46}$$

The fractional operators above can also be defined using Hankel transformation. Let $\varphi \in \mathcal{S}_e$ again.

$$\left(\mathcal{H}_a(-\Delta_a)^{\frac{s}{2}}\varphi\right)(\xi) = |\xi|^s \hat{\varphi}(\xi), \tag{47}$$

see [23], and

$$\left(\mathcal{H}_a(I-\Delta_a)^{\frac{s}{2}}\varphi\right)(\xi) = (1+|\xi|^2)^{\frac{s}{2}} \hat{\varphi}(\xi), \tag{48}$$

see e.g. [13].

Subsequently we need the following observations.

Remark.

Let $\varphi \in \mathcal{S}_e$, $h \in \mathcal{S}_e'$, $0 \le s \le 2$ and let us denote by

$$D_s = (-\Delta_a)^{\frac{s}{2}} \quad \text{or} \quad D_s = (I-\Delta_a)^{\frac{s}{2}}.$$

Then we have

$$\langle D_s\varphi, h\rangle_a = \langle \varphi, D_s h\rangle_a \tag{49}$$

and

$$D_s(\varphi *_a h) = D_s\varphi *_a h = \varphi *_a D_s h. \tag{50}$$

Indeed, let $\varphi, h \in \mathcal{S}_e$. Then (49) follows from (47) or (48) and from the Parseval's formula, see [20, page 20], in a standard way. After this we can naturally extend the operators to distributions by (49).
To derive (50) let $\varphi, h \in \mathcal{S}_e$ again. Taking into account (18), (47) or (48), (50) can be proved by Hankel transform as well. Also taking into account (17), Hankel transform ensures that if $\varphi \in \mathcal{S}_e$, then $D_s T^t \varphi = T^t D_s \varphi$. Thus, in view of (49) we have

$$\langle D_s h, T^t\varphi\rangle_a = \langle h, D_s T^t\varphi\rangle_a = \langle h, T^t D_s\varphi\rangle_a.$$

After the natural extension the first equality can be seen as follows. Let $\psi \in \mathcal{S}_e$.

$$\langle D_s(\varphi *_a h), \psi\rangle_a = \langle \varphi *_a h, D_s\psi\rangle_a = \langle h, \varphi *_a D_s\psi\rangle_a = \langle h, D_s\varphi *_a \psi\rangle_a = \langle D_s\varphi *_a h, \psi\rangle_a.$$

Now we are in position to define the next Bessel B-potential spaces.

Definition 6. *The Bessel B-potential space for $1 \le p \le \infty$, with parameter $\nu > 0$ is*

$$L_a^{\nu,p} := L_a^{\nu,p}(\mathbb{R}_+^n) = \{f : f = G_{a,\nu} *_a g;\ g \in L_a^p\}, \qquad \|f\|_{p,a,\nu} := \|g\|_{p,a}. \tag{51}$$

Remark.
(1) Bessel- or B-potential spaces contain functions that arise as a convolution of a Bessel kernel and a function in L^p. We call these spaces as Bessel B-potential, because here we have Bessel convolution with Bessel kernels.
(2) Direct calculations show that if $g \in C_0^\infty$, then $G_{a,s} *_a g \in \mathcal{S}_e$.

For integers the connection between Bessel-Sobolev and Bessel B-potential spaces is described by the following statement, see [19, Lemma 2.] or [29, Theorem 4.4].

Statement 1. [19, Lemma 2.] *Let $a_i > 0$ $i = 1, \dots, n$ and m be a positive integer, $1 < p < \infty$. Then (with equivalent norms)*

$$W_{\Delta_a}^{m,p} = L_a^{2m,p}.$$

The non-integer case is a little different, cf. [1, Theorem 7.63].

Theorem 3. *Let $1 < p < \infty$, $s > \varepsilon > 0$ and $0 < s < 2$ be arbitrary. Then*

$$L_a^{s+\varepsilon,p} \hookrightarrow W_{\Delta_a}^{\frac{s}{2},p} \hookrightarrow L_a^{s-\varepsilon,p}.$$

Together with Theorem 2 the following proposition implies Theorem 3.

Proposition 1. *Let $1 < p < \infty$, $s > \varepsilon > 0$, $0 < s < 2$ be arbitrary. Let $f \in L_a^{s+\varepsilon,p}$. Then*

$$[f]_{sph,s,p,a} \le c\|f\|_{p,a,s+\varepsilon}. \tag{52}$$

Let $f \in \mathcal{S}_e$. Then there is a $g \in L_a^p$ such that $f = G_{a,s-\varepsilon} *_a g$, and we have

$$\|g\|_{p,a} \le c\|f\|_{W_{\Delta_a}^{\frac{s}{2},p}}. \tag{53}$$

In the inequalities above the constants are independent of f and g.

To prove the above proposition we need the following lemma, see [13] or [27].

Lemma 8. [13, Theorem 3] *For all $\varphi \in \mathcal{S}_e$, $s > 0$ we have*

$$(1-\Delta_a)^{\frac{s}{2}}(G_{a,s} *_a \varphi) = G_{a,s} *_a (1-\Delta_a)^{\frac{s}{2}}\varphi = \varphi. \tag{54}$$

Proof. (of Proposition 1) First we prove (52). Let $f = G_{a,s+\varepsilon} *_a g$. In view of Theorem 2 and the above remark we can assume that $g \in C_0^\infty(\mathbb{R}_+^n)$. Then by (18), (25) and (44) we have

$$\begin{aligned}\mathcal{H}_a(\Delta_{sph,r,a}f)(\xi) &= \left(1 - j_{\frac{n+|a|}{2}-1}(r|\xi|)\right)\frac{1}{(1+|\xi|^2)^{\frac{s+\varepsilon}{2}}}\hat{g}(\xi)\\ &= \mathcal{H}_a\left((\Delta_{sph,r,a}G_{a,s+\varepsilon}) *_a g\right)(\xi).\end{aligned}$$

$$\|\Delta_{sph,r,a}f\|_{p,a} \le \|\Delta_{sph,r,a}G_{a,s+\varepsilon}\|_{1,a}\|g\|_{p,a},$$

and according to (38) we have

$$[f]_{sph,s,p,a} \le \|g\|_{p,a}\left(\int_0^\infty \frac{1}{r^{sp+1}}\|\Delta_{sph,r,a}G_{a,s+\varepsilon}\|_{1,a}^p dr\right)^{\frac{1}{p}} =: \|g\|_{p,a}H(G,a,s,\varepsilon,p).$$

$$\begin{aligned}&H(G,a,s,\varepsilon,p)\\ &\le \left(\int_0^1 \frac{1}{r^{sp+1}}\|\Delta_{sph,r,a}G_{a,s+\varepsilon}\|_{1,a}^p dr\right)^{\frac{1}{p}} + \left(\int_1^\infty \frac{1}{r^{sp+1}}\|\Delta_{sph,r,a}G_{a,s+\varepsilon}\|_{1,a}^p dr\right)^{\frac{1}{p}}\\ &= H_1 + H_2.\end{aligned}$$

By (12)

$$\|\Delta_{sph,r,a}G_{a,s+\varepsilon}\|_{1,a} \le \|G_{a,s+\varepsilon}\|_{1,a} < \infty,$$

so

$$H_2 \le c.$$

Denoting by $t := |x|$ let us recall that by (19) and (21) we have

$$\begin{aligned}&\mathcal{H}_a(\Delta_{sph,r,a}G_{a,s+\varepsilon})\\ &= c(n,a)\mathcal{H}_{\frac{n+|a|}{2}-1}\left(\frac{K_{\frac{n+|a|-(s+\varepsilon)}{2}}(t)}{t^{\frac{n+|a|-(s+\varepsilon)}{2}}} - T^r_{\frac{n+|a|}{2}-1}\frac{K_{\frac{n+|a|-(s+\varepsilon)}{2}}(t)}{t^{\frac{n+|a|-(s+\varepsilon)}{2}}}\right).\end{aligned}$$

Since $G_{a,s+\varepsilon}\in L^1_a(\mathbb{R}^n_+)$ and $G_b(t):=\frac{K_{\frac{n+|a|-(s+\varepsilon)}{2}}(t)}{t^{\frac{n+|a|-(s+\varepsilon)}{2}}}\in L^1_{n+|a|-1}(\mathbb{R}_+)$, by (16) and (21) we have

$$\|\Delta_{sph,r,a}G_{a,s+\varepsilon}(|x|)\|_{1,a}=c_{a,n}$$
$$\times\int_0^\infty\left|\int_0^\pi\left(G_b(t)-G_b\left(\sqrt{t^2+r^2-2rt\cos\vartheta}\right)\right)\sin^{n+|a|-1}\vartheta d\vartheta\right|t^{n+|a|-1}dt$$
$$=c_{a,n}\left(\int_0^{\frac{r}{2}}|\cdot|t^{n+|a|-1}dt+\int_{\frac{r}{2}}^{2r}|\cdot|t^{n+|a|-1}dt+\int_{2r}^\infty|\cdot|t^{n+|a|-1}dt\right)$$
$$=h_1(r)+h_2(r)+h_3(r).$$
$$H_1\le\left(\int_0^1\left(\frac{h_1(r)}{r^{s+\frac1p}}\right)^pdr\right)^{\frac1p}+\left(\int_0^1\left(\frac{h_2(r)}{r^{s+\frac1p}}\right)^pdr\right)^{\frac1p}+\left(\int_0^1\left(\frac{h_3(r)}{r^{s+\frac1p}}\right)^pdr\right)^{\frac1p}.$$

Let $u:=\sqrt{t^2+r^2-2rt\cos\vartheta}$. Since $|r-t|\le u\le r+t$, in the first integral $u>\frac{r}{2}$. In view of (7)

$$\int_0^1\left(\frac{h_1(r)}{r^{s+\frac1p}}\right)^pdr$$
$$\le c(n,a,s)\int_0^1\frac{1}{r^{sp+1}}\left(\int_0^{\frac{r}{2}}\left(\frac{1}{t^{n+|a|-s-\varepsilon}}-\frac{1}{r^{n+|a|-s-\varepsilon}}\right)t^{n+|a|-1}dt\right)^pdr$$
$$\le c(n,a,s)\int_0^1\frac{r^{(s+\varepsilon)p}}{r^{sp+1}}dr=c(n,a,s,\varepsilon)<\infty.$$

According to (9) we have

$$\int_0^1\left(\frac{h_2(r)}{r^{s+\frac1p}}\right)^pdr$$
$$\le c\int_0^1\frac{1}{r^{sp+1}}\left(r\max\left\{\frac{1}{r^{n+|a|-s-\varepsilon}},\right.\right.$$
$$\left.\left.\int_0^\pi G_b\left(\sqrt{t^2+r^2-2rt\cos\vartheta}\right)\sin^{n+|a|-1}\vartheta d\vartheta\right\}r^{n+|a|-1}\right)^pdr.$$

Let $t=(1+\delta)r$, $-\frac12\le\delta\le1$. Recalling the notation above, $u^2=r^2\left(\delta^2+4(1+\delta)\sin^2\frac{\vartheta}{2}\right)\ge r^2\sin^2\vartheta$. Thus by (7) we get

$$\int_0^\pi G_b\left(\sqrt{t^2+r^2-2rt\cos\vartheta}\right)\sin^{n+|a|-1}\vartheta d\vartheta\le c\frac{1}{r^{n+|a|-s-\varepsilon}}\int_0^\pi\frac{\vartheta^{n+|a|-1}}{\vartheta^{n+|a|-s-\varepsilon}}d\vartheta$$
$$=c(s,\varepsilon)\frac{1}{r^{n+|a|-s-\varepsilon}},$$

and so as above we have

$$\int_0^1\left(\frac{h_2(r)}{r^{s+\frac1p}}\right)^pdr\le c\int_0^1\frac{1}{r^{sp+1}}\left(r\frac{1}{r^{n+|a|-s-\varepsilon}}r^{n+|a|-1}\right)^pdr=\int_0^1r^{\varepsilon p-1}dr<\infty.$$

Finally, in view of (8) we have

$$\int_0^1\left(\frac{h_3(r)}{r^{s+\frac1p}}\right)^pdr\le c\int_0^1\frac{1}{r^{sp+1}}\left(\int_{2r}^\infty\frac{e^{-t}}{t^{1-s-\varepsilon}}dt\right)^pdr\le c\int_0^1r^{\varepsilon p-1}dr<\infty.$$

Thus

$$H_1<c,$$

and (52) is proven.

To prove (53) we use (46).
Recalling that $f \in \mathcal{S}_e$, according to Lemma 8 we have

$$f = G_{a,s-\varepsilon} *_a (I-\Delta_a)^{\frac{s-\varepsilon}{2}} f. \tag{55}$$

Thus we have to estimate $\|(I-\Delta_a)^{\frac{s-\varepsilon}{2}} f\|_{p,a}$. In view of (46) we need the following estimation.

$$\left(\int_{\mathbb{R}^n_+} \left| \int_{\mathbb{R}^n_+} (T_a^y f(x) - f(x)) G_{a,-(s-\varepsilon)}(|y|) y^a dy \right|^p x^a dx \right)^{\frac{1}{p}}$$
$$\leq \left\| \int_{|y|\leq 1 \cap \mathbb{R}^n_+} (\cdot) \right\|_{p,a} + \left\| \int_{|y|>1 \cap \mathbb{R}^n_+} (\cdot) \right\|_{p,a} = I + II.$$

(8) and (12) ensure that

$$II \leq c\|f\|_{p,a}.$$

In view of (7) we have

$$I \leq \left\| \int_0^1 \int_{S^n_{1+}} (T^{r\theta} f(x) - f(x)) \theta^a dS(\theta) \frac{1}{r^{1+s-\varepsilon}} dr \right\|_{p,a}$$
$$\leq c(n,a) \left\| \int_0^1 \frac{|\Delta_{sph,r,a} f(x)|}{r^{1+s-\varepsilon}} dr \right\|_{p,a}.$$

Since $(\cdot)^p$ is convex, we have

$$\left(\int_0^1 \frac{|\Delta_{sph,r,a} f(x)|}{r^{s+\frac{\varepsilon}{p-1}} r^{1-\frac{p\varepsilon}{p-1}}} dr\right)^p \leq \left(\int_0^1 \frac{1}{r^{1-\frac{p\varepsilon}{p-1}}}\right)^{p-1} \int_0^1 \frac{|\Delta_{sph,r,a} f(x)|^p}{r^{p\left(s+\frac{\varepsilon}{p-1}\right)} r^{1-\frac{p\varepsilon}{p-1}}} dr,$$

cf. [17, (6.14.1)]. That is

$$I \leq c(n,a,p,\varepsilon) \left(\int_{\mathbb{R}^n_+} \int_0^1 \frac{|\Delta_{sph,r,a} f(x)|^p}{r^{ps+1}} dr x^a dx\right)^{\frac{1}{p}} = c \left(\int_0^1 \frac{\|\Delta_{sph,r,a} f(x)\|^p_{p,a}}{r^{ps+1}} dr\right)^{\frac{1}{p}},$$

so the estimations of I and II together imply (53). $\square$

Proof. (of Theorem 3.) Since $s > 0$, by (7) and (8) $\|f\|_{p,a} \leq c\|f\|_{p,a,s+\varepsilon}$, i.e. if $f \in \mathcal{S}_e$,

$$\|f\|_{p,a,s-\varepsilon} \leq c\|f\|_{W^{\frac{s}{2},p}_{\Delta_a}} \leq c\|f\|_{p,a,s+\varepsilon}. \tag{56}$$

It remains only to prove that $W^{\frac{s}{2},p}_{\Delta_a} \subset L^{s-\varepsilon,p}_a$.
According to theorem 2 and the remark after Definition 51 $\mathcal{S}_e$ is dense in both Banach spaces (with respect to the appropriate norm). If Let $f \in W^{\frac{s}{2},p}_{\Delta_a}$. $\{f_n\} \subset \mathcal{S}_e$ such that $f_n \to f$ in $W^{\frac{s}{2},p}_{\Delta_a}$-norm, then it is also Cauchy in $L^{s-\varepsilon,p}_a$-norm, i.e. there is an $\tilde{f} := \lim_{n\to\infty} f_n$ in $L^{s-\varepsilon,p}_a$. Since $f, \tilde{f} \in L^p_a$ we have $\|f-\tilde{f}\|_{p,a} \leq \|f-f_n\|_{p,a} + \|f_n - f_m\|_{p,a} + \|\tilde{f} - f_m\|_{p,a} \leq c(\|f-f_n\|_{W^{\frac{s}{2},p}_{\Delta_a}} + \|f_n - f_m\|_{p,a,s-\varepsilon} + \|\tilde{f} - f_m\|_{p,a,s-\varepsilon}$.
Thus $f = \tilde{f}$ a.e.. $\square$

6. Removable sets

The previous sections were motivated by this last one, that is to clarify removability of compact sets with respect to fractional Bessel differential operators.

Definition 7. *We say that a compact set $K \subset \mathbb{R}_+^n$ is removable with respect to the partial differential operator L in $L_a^p(\mathbb{R}_+^n)$ if for any $u \in L_a^p(\mathbb{R}_+^n \setminus K)$, $Lu = 0$ on $\mathbb{R}_+^n \setminus K$ there is a $\tilde{u} \in L_a^p(\mathbb{R}_+^n)$ such that $\tilde{u}|_{\mathbb{R}_+^n \setminus K} = u$ and $L\tilde{u} = 0$ on $\mathbb{R}_+^n$.*

As it is mentioned in the introduction, potential theoretic characterization of removability dates back to Carleson. A fundamental theorem in the classical case is the following one, see [2, Theorem 2.7.4].

Theorem A. *Let $K \subset \mathbb{R}^n$ be compact and let $\mathcal{L}$ be an elliptic linear partial differential operator of order less than n with constant coefficients. Then K is removable for $\mathcal{L}$ in L^p, $1 < p < \infty$, if $N_{\alpha,p'}(K) = 0$, and is not removable if $C_{\alpha,p'}(K) > 0$.*

There is a series of recent papers on potential theoretic characterization of removability with respect to different fractional operators, see e.g. [11] and the references therein.

We start with the definition of different capacities (cf. [2, Definitions 2.3.3 and 2.7.1]), which allows the description of removability in certain L^p spaces. Recall that $\mathcal{S}_e \subset L_a^{p,s}$ is dense.

Definition 8. *Let $K \subset \mathbb{R}_+^n$ be compact. $U \subset \mathbb{R}_+^n$ stands for a neighbourhood of K. The Bessel capacity of a compact set is*

$$C_{p,a,s}(K) := \inf\left\{\|f\|_{p,a,s}^p : f = G_{a,s} *_a g \in \mathcal{S}_e,\ g \geq 0;\ f(x) \geq 1,\ \forall x \in K\right\}.$$

The modified Bessel capacity is defined as follows.

$$N_{p,a,s}(K) := \inf\left\{\|f\|_{p,a,s}^p : f = G_{a,s} *_a g \in \mathcal{S}_e,\right.$$
$$\left. g \geq 0;\ \exists\ U,\ K \subset U,\ f(x) = 1,\ \forall x \in U\right\}$$

Remark.
- It can be readily seen that if $N_{p,a,s}(K) = 0$, then $\lambda(K) = 0$, see e.g. [19].
- According to [2, Theorem 2.5.1] we have

$$C_{p,a,s}(K)^{\frac{1}{p}} = \sup\left\{\mu_a(K) : \|G_{a,s} *_a \mu\|_{p',a} \leq 1\right\}, \tag{57}$$

where μ is a positive measure supported on K, $\mu_a(K) := \int_K x^a d\mu(x)$.
- Obviously, $C_{p,a,s}(K) \leq N_{p,a,s}(K)$.

Notation.
Let $0 < s_k < 2$ and c_k be real numbers ($k = 0, \dots, m$). We define the next fractional partial differential operator as follows.

$$Lu := \sum_{k=1}^{m} c_k (I - \Delta_a)^{\frac{s_k}{2}} u. \tag{58}$$

In accordance with Theorem A, in fractional Bessel case we have the following theorem.

Theorem 4. *Let $1 < p < \infty$, L be as in (58) and $K \subset \mathbb{R}^n_+$ be compact. Let $2 > s > \max_{k=1}^m s_k$. If $N_{p',a,s}(K) = 0$, then K is removable with respect to L in $L^p_a(\mathbb{R}^n_+)$. If $m = 1$, and K is removable with respect to $L = (I - \Delta_a)^{\frac{s}{2}}$ in $L^p_a(\mathbb{R}^n_+)$, then $C_{p',a,s}(K) = 0$.*

Proof. Let $N_{p',a,s}(K) = 0$. Then for all $\varepsilon > 0$ there is a $\varphi = G_{a,s} *_a g \in \mathcal{S}_e$ such that $\|\varphi\|_{p',a,s} < \varepsilon$ and $\varphi \equiv 1$ on some neighbourhood U of K, cf. (51). Let O be an open set such that $K \subset U \subset O \subset \mathbb{R}^n_+$, and let $\psi \in C_0^\infty(O)$. We may assume that the distance between the boundaries of O and $\mathbb{R}^n_+$ is positive. Then $(1-\varphi)\psi \in C_0^\infty(O \setminus K)$. Let $u \in \mathcal{S}'_e \cap L^p_a(O \setminus K)$ such that $Lu = 0$ on $O \setminus K$. In view of (49) we have

$$\langle Lu, (1-\varphi)\psi\rangle_a = \langle u, L(1-\varphi)\psi\rangle_a = \langle u, L\psi\rangle_a - \langle u, L\varphi\psi\rangle_a = 0.$$

Thus by the remark above we have

$$|\langle u, L\psi\rangle_a| = |\langle u, L\varphi\psi\rangle_a| \le \|u\|_{p,a,O}\|L\varphi\psi\|_{p',a}.$$

By linearity it is enough to deal with a single term. Since

$$L_{s_k}\varphi\psi := (I - \Delta_a)^{\frac{s_k}{2}}\varphi\psi = L_{s_k}((G_{a,s} *_a g)\psi),$$

by (49) we have

$$\sup_{\tau\in\mathcal{S}_e, \|\tau\|_{p,a}\le 1} \langle G_{a,s} *_a g, \psi L_{s_k}\tau\rangle_a$$

$$\le \|\psi\|_\infty \sup_{\tau\in\mathcal{S}_e, \|\tau\|_{p,a}\le 1} \langle G_{a,s} *_a g, |(I-\Delta_a)^{\frac{s_k}{2}}\tau|\rangle_a$$

$$= c \sup_{\tau\in\mathcal{S}_e, \|\tau\|_{p,a}\le 1} \langle (I-\Delta_a)^{\frac{s_k}{2}}(G_{a,s} *_a g), \tau\rangle_a$$

$$= c \sup_{\tau\in\mathcal{S}_e, \|\tau\|_{p,a}\le 1} \langle G_{a,s-s_k} *_a g, \tau\rangle_a \le c\|g\|_{p',a} \le c\varepsilon.$$

The last but one step can be seen by Hankel transform again. Indeed, if $g \in C_0^\infty$, the inversion formula, (16) can be applied on both sides, moreover by (44), (48) and (18) $\mathcal{H}_a\left((I-\Delta_a)^{\frac{s_k}{2}}(G_{a,s} *_a g)\right)(\xi) = \mathcal{H}_a\left(G_{a,s-s_k} *_a g\right)(\xi) = \frac{1}{(1+|\xi|^2)^{\frac{s-s_k}{2}}}\hat{g}(\xi)$. Since $s - s_k > 0$, in view of (43), $G_{a,s-s_k} \in L^1_a(\mathbb{R}^n_+)$.

Let $m = 1$. Assume that $C_{p',a,s}(K) > 0$. In view of (57) there is a positive measure μ supported on K such that $G_{a,s} *_a \mu \in L^p_a(\mathbb{R}^n_+)$. Together with (14) and (50), Lemma 8 implies that the fundamental solution of the operator $(1-\Delta_a)^{\frac{s}{2}}$ is $G_{a,s} \in \mathcal{S}'_e$. Then $L(G_{a,s} *_a \mu)$ is supported on K, i.e. K is not removable. □

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Both authors:
Department of Analysis and Operations Research,
Institute of Mathematics,
Budapest University of Technology and Economics
Műegyetem rkp. 3., H-1111 Budapest, Hungary.

Á. P. Horváth: g.horvath.agota@renyi.hu
M. Chegaar: mouna.chegaar@edu.bme.hu