QUARTIC CURVES IN THE QUINTIC DEL PEZZO THREEFOLD

KIRYONG CHUNG, JAEHYUN KIM, AND JEONG-SEOP KIM

ABSTRACT. In this paper, we prove that the Hilbert scheme $\mathbf{H}_4(X_5)$ of rational quartic curves on the quintic del Pezzo threefold X_5 is isomorphic to a Grassmannian bundle over the Hilbert scheme of lines on X_5. In particular, $\mathbf{H}_4(X_5)$ is smooth and irreducible. Our approach builds upon the geometry of rational quartic curves on X_5 studied by Fanelli–Gruson–Perrin in their work on the moduli space of stable maps to X_5.

1. INTRODUCTION AND RESULTS

We work over the field $\mathbb{C}$ of complex numbers.

1.1. **Related works and the result.** Fano varieties are covered by families of rational curves, which play an essential role in understanding the geometry of the underlying varieties. For example, lines and conics are used in Iskovskikh's classification of Fano threefolds [IP99], and the moduli space of lines is used by Clemens and Griffiths to disprove the rationality of cubic threefolds [CG72]. On the other hand, the geometry of the moduli spaces of rational curves is interesting in its own right, providing new examples of intriguing varieties (cf. [BD85]), and their compactifications draw significant attention in the context of enumerative geometry (cf. [FP97]).

Let X be a projective variety with fixed embedding $X \subset \mathbb{P}^r$ and $\mathbf{H}_d(X)$ be the Hilbert scheme parametrizing curves C in X with Hilbert polynomial $\chi(\mathcal{O}_C(m)) = dm + 1$. For smooth Fano threefolds X with minimal Betti numbers $b_2(X) = 1$ and $b_3(X) = 0$, some studies have been done for rational curves on X with lower degree d. The geometry of various compactifications of the moduli of rational curves on Fano varieties X has been extensively studied by many researchers, including the first-named author — after studies of various compactifications of the space of twisted cubics in the projective space $\mathbb{P}^3$ [PS85, VX02], their geometries were compared in [CK11]. The case of $X = \mathbb{P}^3$ and $d = 4$ is also studied in [CCM16]. For studies on the quadric threefold Q^3, we refer not only to [CHY23] but also to [Per02] in the context of rational curves on homogeneous varieties, and to [HRS04] in the context of rational curves on hypersurfaces. In addition, we may refer to [Ili94, San14, CHL18, Chu23] for the quintic del Pezzo threefold X_5, and to [MU83, Kuz97] for the prime Fano threefolds V_{22} of degree $(-K_{V_{22}})^3 = 22$. This paper is

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devoted to the study of the case where $X = X_5$ and $d = 4$, a parallel continuation of the authors' previous work on the case where X is the Mukai-Umemura variety U_{22} and $d = 4$ [CKK25]. In this paper, contrary to [CKK25], which only describes the local geometry, we succeed in describing the global geometry of a space of rational curves as follows.

Theorem 1.1. *Let $\mathbf{H}_4(X_5)$ be the Hilbert scheme that parametrizes curves C on the quintic del Pezzo threefold X_5 with Hilbert polynomial $\chi(\mathcal{O}_C(m)) = 4m + 1$. Then $\mathbf{H}_4(X_5)$ is isomorphic to a $\mathrm{Gr}(3, 5)$-bundle over $\mathbb{P}^2$.*

This irreducibility result for $X = X_5$ contrasts with the result for $X = \mathbb{P}^3$, as $\mathbf{H}_4(\mathbb{P}^3)$ is known to have several irreducible components [CN12, Theorem 4.9]. Meanwhile, the irreducibility is natural in light of the authors' previous result [CKK25] on the smoothness of $\mathbf{H}_4(U_{22})$ for $X = U_{22}$, as the geometry of the underlying variety X becomes more complicated, the space $\mathbf{H}_d(X)$ of rational curves tends to simplify.

From Prokhorov's classification of hyperplane sections of X_5 [Pro91, Section 2] together with Fanelli-Gruson-Perrin's description of a Kontsevich moduli space by [FGP19, Section 4.1], we observe that all quartic rational curves arise as quartic curves contained in a hyperplane section of X_5, residual to a line in the same hyperplane section. Consequently, we can generate all residual quartic curves as ideal quotients of the ideals of the linear sections by those of lines. Applying this fact, we construct an *injective* morphism from a Grassmannian $\mathrm{Gr}(3, 5)$-bundle over $\mathbf{H}_1(X_5) = \mathbb{P}^2$ to the Hilbert scheme $\mathbf{H}_4(X_5)$ (Proposition 4.5). This morphism turns out to be an isomorphism, as we show that $\mathbf{H}_4(X_5)$ is a smooth, irreducible variety of dimension 8 (Proposition 5.3 and Proposition 5.7). A key ingredient in the proof of irreducibility is to use a result of Lehmann–Tanimoto [LT19, Theorem 1.5]. They prove that there is a unique irreducible component of the space of stable maps birational onto their image. By analyzing the component of multiple curves, we conclude that the Hilbert scheme $\mathbf{H}_4(X_5)$ is irreducible.

For further work, the authors expect a similar study to be conducted on higher dimensional smooth Fano varieties, such as other linear sections of the Grassmannian $\mathrm{Gr}(2, 5)$. Additionally, as an application of this paper, the authors expect it to serve as a foundation for computing the Donaldson–Thomas invariants (cf. [CT21, CLW24]).

1.2. **Stream of the paper.** This paper is organized as follows. In Section 2, we review the construction of X_5 by comparing its Plücker embedding and SL_2-orbit coordinates. In Section 3, we summarize previous results on the Hilbert scheme $\mathbf{H}_d(X_5)$ for $d \leq 3$, adding additional observations regarding the $\mathbb{C}^*$-actions on X_5. In Section 4, after investigating the geometry of curves of genus $g \leq 1$ in X_5, we construct an injective morphism between $\mathbf{H}_4(X_5)$ and a Grassmannian bundle over $\mathbb{P}^2$. In Section 5, we prove that the prescribed injective morphism is an isomorphism, which is our main result.

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2. SL_2-INVARIANT PRESENTATIONS OF THE DEL PEZZO VARIETY

In this section, we review the two well-known constructions of the quintic del Pezzo threefold and their relation.

2.1. **Plücker v.s. orbit coordinates of X_5.** Let X_5 be the quintic del Pezzo 3-fold with SL_2-invariant setting. We present X_5 in two ways (That is, the Plücker coordinates and a closure of SL_2-orbits) and find its relations explicitly. For the presentation of the Plücker coordinates of X_5, let us denote the vector space $V_5 = \mathrm{Sym}^4\mathbb{C}^2 = \langle e_4, e_2, e_0, e_{-2}, e_{-4}\rangle$ with weights $e_k = -k$ for $k \in \{\pm 4, \pm 2, 0\}$. There exists a unique SL_2-invariant 7-dimensional subspace in

$$\wedge^2 V_5 \cong \mathrm{Sym}^6\mathbb{C}^2 \oplus \mathrm{Sym}^2\mathbb{C}^2.$$

Let us denote p_{ij} by the Plücker coordinates of $\mathbb{P}(\wedge^2 V_5)$ where $p_{ij}(e_k \wedge e_l) = \delta_{ik}\delta_{jl}$ for $i < j$, $k < l$. The following were taken in [CS16, Corollary 5.5.11 and Lemma 5.5.12] (cf. [San14, Section 2.1]). The net of skew-symmetric forms in $\wedge^2 V_5^*$ is defined by

$$3e_2^* \wedge e_0^* - e_4^* \wedge e_{-2}^*,\ 2e_2^* \wedge e_{-2}^* - e_4^* \wedge e_{-4}^*, 3e_0^* \wedge e_{-2}^* - e_2^* \wedge e_{-4}^*,$$

where the operator $f \in \mathfrak{sl}_2(\mathbb{C})$ acts on this subspace in a standard way. Hence, the del Pezzo variety X_5 in $\mathbb{P}(\wedge^2 V_5) = \mathbb{P}^9$ is defined by the equations:

$$I_{\mathrm{Gr}(2,V_5)/\mathbb{P}^9} + \langle 3p_{2,0} - p_{4,-2}, 2p_{2,-2} - p_{4,-4}, 3p_{0,-2} - p_{2,-4}\rangle. \tag{1}$$

Lemma 2.1. *The unique SL_2-invariant 7-dimensional subspace in $\wedge^2 V_5$ is generated by*

$$e_4 \wedge e_2,\ e_4 \wedge e_0,\ e_2 \wedge e_0 + 3e_4 \wedge e_{-2},\ e_2 \wedge e_{-2} + 2e_4 \wedge e_{-4},\ e_0 \wedge e_{-2} + 3e_2 \wedge e_{-4},\ e_0 \wedge e_{-4}, e_{-2} \wedge e_{-4}.$$

Proof. Note that the operation f acts on the dual space $V_5^{**} = V_5$ by

$$f{\cdot}e_{-4} = e_{-2},\ f{\cdot}e_{-2} = 2e_0,\ f{\cdot}e_0 = 3e_2,\ f{\cdot}e_2 = 4e_4.$$

By acting the operator f successively to the obvious highest weight $e_{-2} \wedge e_{-4}$, we obtain the result. □

On the other hand, let us denote by $\mathbf{R}_n = \mathbb{C}[x, y]_n$ the vector space of degree n homogeneous polynomials with respect to x, y. For an element $g(x, y) \in \mathbf{R}_n$, the coefficient a_i of

$$g(x, y) = \sum_{k=0}^{n} a_{2k-n} \binom{n}{k} x^k y^{n-k}$$

is taken as the homogeneous coordinate of $\mathbb{P}^n$. Let $f(x, y) = xy(x^4 - y^4)$ be a degree six homogeneous polynomial with distinct roots. It is well-known that the quintic del Pezzo variety X_5 is defined by the closure of the orbit of the closed point $f(x,y)$ by the action $\mathrm{SL}_2(\mathbb{C})$ ([MU83]):

$$X_5 = \overline{\mathrm{SL}_2(\mathbb{C}) \cdot f(x,y)} \subset \mathbb{P}(\mathbf{R}_6).$$

In this case, the defining equations of X_5 are given by five quadric polynomials:

$$\begin{aligned}&\langle a_6a_{-2} - 4a_4a_0 + 3a_2^2,\ a_6a_{-4} - 3a_4a_{-2} + 2a_2a_0,\ a_6a_{-6} - 9a_2a_{-2} + 8a_0^2,\\ &a_4a_{-6} - 3a_2a_{-4} + 2a_0a_{-2},\ a_2a_{-6} - 4a_0a_{-4} + 3a_{-2}^2\rangle.\end{aligned} \tag{2}$$

Proposition 2.2. *The Plücker coordinates and SL_2-orbit coordinates of the del Pezzo threefold X_5 defined in (1) and (2), respectively, are related by*

$$p_{4,2} = a_6,\ p_{4,0} = 2a_4,\ p_{2,0} = a_2,\ p_{4,-2} = 3a_2,\ p_{2,-2} = 2a_0,$$
$$p_{4,-4} = 4a_0,\ p_{0,-2} = a_{-2},\ p_{2,-4} = 3a_{-2},\ p_{0,-4} = 2a_{-4},\ p_{-2,-4} = a_{-6}.$$

That is, there is a linear embedding of $\mathbb{P}^6 = \mathbb{P}(\mathrm{Sym}^6\mathbb{C}^2)$ into the space $\mathbb{P}(\wedge^2 V_5)$ such that the quintic del Pezzo variety defined by (2) corresponds to the variety defined by (1).

Proof. Let w_i be the standard dual vector of a_i. Let us associate the ordinates of $\mathrm{Sym}^6\mathbb{C}^2$ to the vector space $\wedge^2 V_5$ in the following way (cf. Lemma 2.1):

$$w_6 \mapsto e_4 \wedge e_2,\ w_4 \mapsto 2e_4 \wedge e_0,\ w_2 \mapsto e_2 \wedge e_0 + 3e_4 \wedge e_{-2},\ w_0 \mapsto 2(e_2 \wedge e_{-2} + 2e_4 \wedge e_{-4}),$$
$$w_{-2} \mapsto e_0 \wedge e_{-2} + 3e_2 \wedge e_{-4},\ w_{-4} \mapsto 2e_0 \wedge e_{-4},\ w_{-6} \mapsto e_{-2} \wedge e_{-4}.$$

Then one can easily check that the defining equation (1) has been changed into (2). □

3. Rational curves of degree ≤ 3 in X_5

In this section, we present the $\mathbb{C}^*$-fixed rational curves of degree $d \leq 3$ in X_5 by using several Schubert varieties of $\mathrm{Gr}(2, V_5)$. From now on, let us fix the codimension three linear subspace P, which is defined by the linear forms in (1). That is, $X_5 = \mathrm{Gr}(2, V_5) \cap P$.

3.1. **A torus action on Grassmannian.** Since the following easy lemma will be used several times, we put the details of the proof here for the convenience of readers. For a vector space V, a vector $w \in \wedge^2 V$ is called *decomposable* if $w = v_1 \wedge v_2$ for some $v_1, v_2 \in V$ (equivalently, $\mathbb{P}(w) \in \mathrm{Gr}(2, V)$). A vector w is decomposable if and only if $w \wedge w = 0 \in \wedge^4 V$.

Lemma 3.1. *Let V be the finite-dimensional vector space with the diagonal $\mathbb{C}^*$-action of distinct weights. Then the fixed loci of the Grassmannian variety $\mathrm{Gr}(2, V)$ are isolated whenever each weight occurs with multiplicity at most two among the Plücker coordinates in $\mathbb{P}(\wedge^2 V)$.*

Proof. By the assumption, the fixed loci in $\mathbb{P}(\wedge^2 V)$ are coordinate points when the weight in $\wedge^2 V$ is unique. If the weight of two vectors $e_i \wedge e_j$ and $e_k \wedge e_l$, $\{i,j\} \cap \{k,l\} = \emptyset$ is the same one, then any vector $ue_i \wedge e_j + ve_k \wedge e_l$, $(u,v) \in \mathbb{C}^2$ is a $\mathbb{C}^*$-fixed one. However, the line $\mathbb{P}(ue_i \wedge e_j + ve_k \wedge e_l)$ intersects with $\mathrm{Gr}(2, V)$ at two points $\mathbb{P}(e_i \wedge e_j)$, $\mathbb{P}(e_k \wedge e_l)$ because

$w = ue_i \wedge e_j + ve_k \wedge e_l$ is a decomposable vector if and only if $u = 0$ or $v = 0$, and thus the fixed loci are isolated. $\square$

3.2. **Fixed lines in X_5.** For an embedding of a smooth projective variety $X \subset \mathbb{P}^r$, let $\mathbf{H}_d(X)$ be the Hilbert scheme parametrizing curves C in X with Hilbert polynomial

$$\chi(\mathcal{O}_C(m)) = dm + 1.$$

It is known that the Hilbert scheme of lines in $\mathrm{Gr}(2, V_5)$ is isomorphic to $\mathbf{H}_1(\mathrm{Gr}(2, V_5)) \cong \mathrm{Gr}(1, 3, V_5)$ by [Har95, Exercise 6.9]. A line in $\mathrm{Gr}(2, V_5)$ is determined by a pencil of lines passing through a point (so-called *vertex*). Also, under the natural embedding $\mathbf{H}_1(X_5) \subset \mathbf{H}_1(\mathrm{Gr}(2, V_5))$, we have a $\mathbb{C}^*$-equivariant projection map

$$p : \mathbf{H}_1(X_5) \subset \mathbf{H}_1(\mathrm{Gr}(2, V_5)) \cong \mathrm{Gr}(1, 3, V_5) \to \mathrm{Gr}(1, V_5)$$

which associates a line to its *vertex*. Clearly, the fixed locus of $\mathrm{Gr}(1, V_5)$ is just the coordinate points. For each point $\mathbb{P}(e_i)$, $i \in \{\pm 4, \pm 2, 0\}$, the fixed point of the fiber $p^{-1}(\mathbb{P}(e_i))$ determines the fixed loci of $\mathbf{H}_1(X_5)$. However, the fiber of the map p is isomorphic to $\mathrm{Gr}(2, V_5/\langle e_i \rangle)$. Note that the weights of the quotient space $V_5/\langle e_i \rangle$ are distinct from each other. Thus by Lemma 3.1, the $\mathbb{C}^*$-fixed lines in $\mathrm{Gr}(2, V_5)$ are of the form:

$$\{e_i \wedge (ue_s + ve_t)\}, [u : v] \in \mathbb{P}^1$$

for $s, t \in \{\pm 4, \pm 2, 0\} \setminus \{i\}$. By plugging the above lines into the defining relation (1) of the del Pezzo variety, we can check that there exist exactly three fixed lines in X_5 as follows.

$$e_4 \wedge (ue_2 + ve_0), e_2 \wedge (ue_0 + ve_{-2}), e_0 \wedge (ue_{-2} + ve_{-4}), [u : v] \in \mathbb{P}^1. \tag{3}$$

Remark 3.2. Under SL_2-orbit coordinates of the del Pezzo variety X_5, lines in X_5 are given by ([KPS18, Lemma 5.1.4])

$$L_{uv} = \{uv(su^4 - tv^4)\}_{[s:t] \in \mathbb{P}^1},\ u, v \in \mathbb{P}(\mathbf{R}_1), u \neq v$$

or

$$L_{u^2} = \{u^5(sx + ty)\}_{[s:t] \in \mathbb{P}^1},\ u \in \mathbb{P}(\mathbf{R}_1).$$

Hence, the fixed lines are L_{xy}, L_{x^2}, L_{y^2}, which match that of (3).

3.3. **Fixed conics in X_5.** The geometry of the Hilbert scheme of conics in the Grassmannian variety $\mathrm{Gr}(2, V_n)$, $\dim V_n = n$ has been intensively studied in [HT15, CHL18, CM17]. The key observation is to consider the linear spanning of a conic in two different ways: The linear spanning of a conic in the Plücker embedding $\mathbb{P}(\wedge^2 V_n)$ and the spanning of the moduli points of a conic in $\mathbb{P}(V_n)$. For the del Pezzo threefold X_5, by considering the spanning as the moduli meaning, there exists an isomorphism

$$\mathbf{H}_2(X_5) \cong \mathrm{Gr}(4, V_5),$$

where for a subvector space $V_4 < V_5$, the conic is given by the intersection

$$\mathrm{Gr}(2, V_4) \cap P \subset \mathrm{Gr}(2, V_5) \cap P = X_5.$$

For an explicit description of the above isomorphism, see [CHL18, Proposition 4.3]. Hence, the $\mathbb{C}^*$-fixed conic is represented by the five vector spaces

$$W_4 = \{e_2, e_0, e_{-2}, e_{-4}\},\ W_2 = \{e_4, e_0, e_{-2}, e_{-4}\},$$
$$W_0 = \{e_4, e_2, e_{-2}, e_{-4}\},\ W_{-2} = \{e_4, e_2, e_0, e_{-4}\},\ W_{-4} = \{e_4, e_2, e_0, e_{-2}\}.$$

If $V_4 = W_4$, the defining equation of the conic in X_5 is

$$I_{\mathrm{Gr}(2,W_4)\cap P} = I_{X_5/\mathbb{P}^9} + \langle p_{4,j} \mid j \in \{2, 0, -2, -4\}\rangle.$$

In terms of SL_2-orbit coordinates, the conic is defined by

$$\langle a_6, a_4, a_2, a_0, a_{-2}^2\rangle.$$

The other four cases are given by the following ones.

$$\langle a_6, a_2, a_0, a_{-2}, a_4a_{-6}\rangle,\ \langle a_4, a_2, 8a_0^2 + a_6a_{-6}, a_{-2}, a_{-4}\rangle,$$
$$\langle a_6a_{-4}, a_2, a_0, a_{-2}, a_{-6}\rangle,\ \langle a_2^2, a_0, a_{-2}, a_{-4}, a_{-6}\rangle.$$

3.4. **Fixed twisted cubics in X_5.** In [San14, Proposition 2.46 and Remark 2.47], the author proves that the Hilbert scheme of twisted cubic curves is isomorphic to

$$\mathbf{H}_3(X_5) \cong \mathrm{Gr}(2, V_5).$$

Furthermore, he described the universal resolution of twisted cubics in X_5. By using a Schubert variety, the above isomorphism can be described by finding the defining equation of the twisted cubics (cf. [Chu23, Proposition 3.3]). In detail, let $\sigma_1(l)$ be the Schubert variety of lines meeting a fixed line l in $\mathbb{P}(V_5)$. The variety $\sigma_1(l)$ is a degree three, four-dimensional variety in $\mathbb{P}(\wedge^2 V_5)$. Hence, by cutting out by the codimension 3 hyperplane P, we obtain a twisted cubic curve in X_5.

Since the weights of V_5 are different (Lemma 3.1), there are ten $\mathbb{C}^*$-fixed twisted cubics in X_5. For instance, let $l = \langle e_4, e_2\rangle$ be the line in $\mathbb{P}(V_5)$. Then the lines meeting with the line l would be generically described by the decomposable vectors

$$(e_4 + t_0e_2) \wedge (e_2 + t_1e_0 + t_2e_{-2} + t_3e_{-4}) \tag{4}$$

for $t_i \in \mathbb{C}^*$ in $\mathbb{P}(\wedge^2 V_5)$. Therefore, the variety $\sigma_1(l)$ is the closure of the image of the map $(\mathbb{C}^*)^4 \to \mathbb{P}(\wedge^2 V_5)$ given by equation (4). That is,

$$I_{\sigma_1(l)} = \langle \det \begin{bmatrix} p_{4,0} & p_{4,-2} & p_{4,-4} \\ p_{2,0} & p_{2,-2} & p_{2,-4} \end{bmatrix}, p_{i,j}\rangle,\ i, j \in \{0, -2, -4\}.$$

After cutting out by the codimension three linear subspace P in (1), we obtain the $\mathbb{C}^*$-fixed twisted cubic C_l in X_5, which presents the point $[l]$. In terms of orbit coordinates, the defining equation of C_l is given by

$$I_{C_l/\mathbb{P}^6} = \langle a_{-6}, a_{-4}, a_{-2}, a_0^2, a_2a_0, 3a_2^2 - 4a_4a_0\rangle.$$

In the same manner, we have nine other $\mathbb{C}^*$-fixed twisted cubics in X_5 in Table 1 and their configurations are presented in Figure 1. We remark that the multiple structure of a twisted cubic curve of a given reduced curve is unique in each case ([Chu23, Lemma 3.2 and Lemma 3.5]).

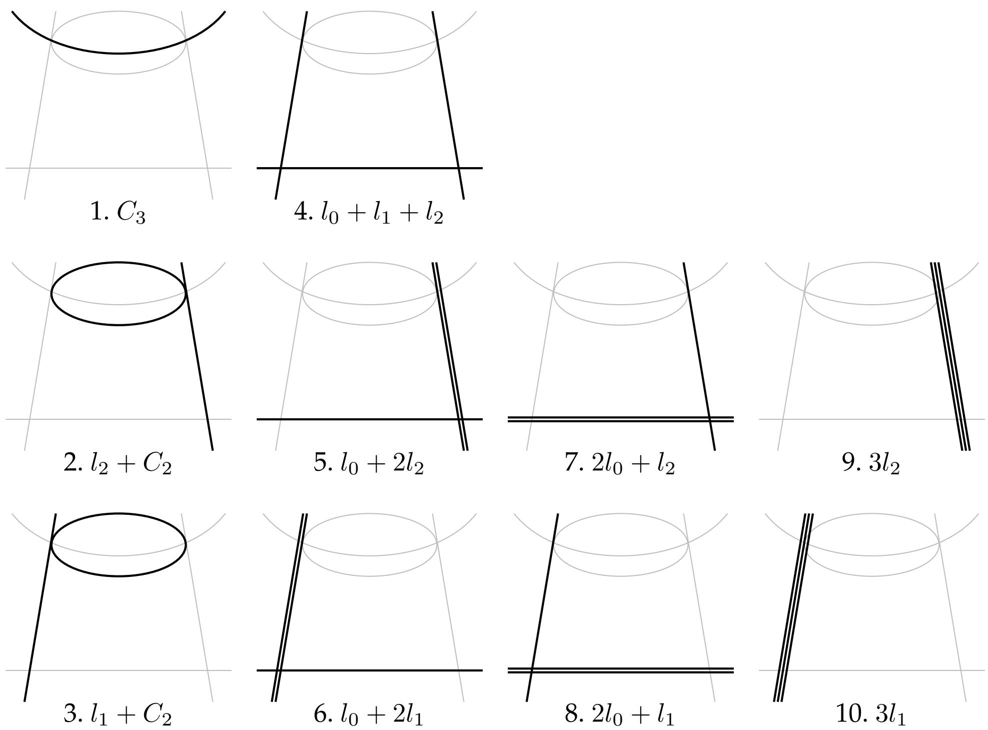
TABLE 1. $\mathbb{C}^*$-fixed twisted cubics

No.	l	$I_{C_l/\mathbb{P}^6}$	
1	$\langle e_2, e_{-2}\rangle$	$\langle a_{-4}, a_0, a_4, 3a_{-2}^2 + a_2a_{-6}, 9a_2a_{-2} - a_6a_{-6}, 3a_2^2 + a_6a_{-2}\rangle$	C_3
2	$\langle e_2, e_{-4}\rangle$	$\langle a_{-2}, a_2, a_4, a_0a_{-4}, a_6a_{-4}, 8a_0^2 + a_6a_{-6}\rangle$	$l_2 + C_2$
3	$\langle e_4, e_{-2}\rangle$	$\langle a_{-4}, a_{-2}, a_2, a_4a_{-6}, a_4a_0, 8a_0^2 + a_6a_{-6}\rangle$	$l_1 + C_2$
4	$\langle e_4, e_{-4}\rangle$	$\langle a_{-2}, a_0, a_2, a_4a_{-6}, a_6a_{-6}, a_6a_{-4}\rangle$	$l_0 + l_1 + l_2$
5	$\langle e_0, e_{-4}\rangle$	$\langle a_0, a_2, a_6, a_4a_{-6}, a_4a_{-2}, a_{-2}^2\rangle$	$l_0 + 2l_2$
6	$\langle e_4, e_0\rangle$	$\langle a_{-6}, a_{-2}, a_0, a_2a_{-4}, a_6a_{-4}, a_2^2\rangle$	$l_0 + 2l_1$
7	$\langle e_0, e_{-2}\rangle$	$\langle a_{-2}, a_0, a_6, a_2a_{-6}, a_2^2, 3a_2a_{-4} - a_4a_{-6}\rangle$	$2l_0 + l_2$
8	$\langle e_2, e_0\rangle$	$\langle a_{-6}, a_0, a_2, a_{-2}^2, a_6a_{-2}, 3a_4a_{-2} - a_6a_{-4}\rangle$	$2l_0 + l_1$
9	$\langle e_{-2}, e_{-4}\rangle$	$\langle a_2, a_4, a_6, a_0a_{-2}, a_0^2, 3a_{-2}^2 - 4a_0a_{-4}\rangle$	$3l_2$
10	$\langle e_4, e_2\rangle$	$\langle a_{-6}, a_{-4}, a_{-2}, a_0^2, a_2a_0, 3a_2^2 - 4a_4a_0\rangle$	$3l_1$

4. RATIONAL AND ELLIPTIC QUARTIC CURVES IN X_5

As mentioned in the proof of [FGP19, Proposition 4.11], for general $\mathbb{P}^4 \subset \mathbb{P}^6 = \mathbb{P}(\mathrm{Sym}^6\mathbb{C}^2)$, the (scheme-theoretic) intersection part $X_5 \cap \mathbb{P}^4 = E_5$ is an elliptic quintic curve in X_5 by adjunction formula. However, if we choose every general linear subspace $\mathbb{P}^4$ containing a line L in X_5, then the intersection curve E_5 is a union of L and a rational quartic curve C_4. Note that an irreducible rational quartic curve C_4 has a unique secant line in X_5 ([FGP19, Lemma 4.14]). In this section, we will describe the Hilbert scheme of rational quartic curves in X_5 by extending this geometric process.

FIGURE 1. Configuration of $\mathbb{C}^*$-fixed twisted cubic curves



4.1. **Quartic curves via residual curve.** Let us start with a simple observation of the hyperplane section of quintic del Pezzo varieties.

Lemma 4.1 ([Pro91, Section 2])**.** *The (scheme-theoretical) hyperplane section $X_5 \cap H$ of any hyperplane H in $\mathbb{P}^6$ is a non-degenerate, irreducible (possibly, singular) quintic del Pezzo surface.*

Proof. Since X_5 is non-degenerate, the intersection $F = X_5 \cap H$ is a surface. Note that $[F]$ is primitive in $\mathrm{Pic}(X_5)$. Because X_5 is smooth, an irreducible component F_1 of F becomes an element of $\mathrm{Pic}(X_5)$, and hence, $[F_1] = [F]$. That is, $X_5 \cap H$ is irreducible. The non-degeneracy of $X_5 \cap H$ follows from Lemma 8.1 in [BL13]. $\square$

Let us describe the construction of a rational quartic curve using a sheaf-theoretical method. For a line $L \subset X_5 = X$, let s_1, $s_2 \in \mathrm{H}^0(I_{L/X_5}(1))$ be the sub-linear section of $\mathrm{H}^0(\mathcal{O}_X(1))$ such that $s_1 \wedge s_2 \neq 0$. Then it gives rise to an exact sequence

$$0 \to \mathcal{O}_{X_5}(-2) \stackrel{\psi=(s_1,s_2)}{\longrightarrow} \mathcal{O}_{X_5}(-1) \oplus \mathcal{O}_{X_5}(-1) \to \mathrm{coker}(\psi) \to 0. \tag{5}$$

The scheme theoretic intersection $E_5 = X_5 \cap V(s_1) \cap V(s_2)$ in the irreducible surface $X_5 \cap V(s_1)$ is a curve by Lemma 4.1. Therefore, by computing Hilbert polynomials of each term in (5), we know that $\mathrm{coker}(\psi) \cong I_{E_5/X_5}$, $\chi(\mathcal{O}_{E_5}(m)) = 5m$. The curve E_5 is connected by [FH79] and clearly contains the line L by its construction.

Proposition 4.2. *Under the above assumption, for any elliptic quintic curve E_5 in X_5 given a codimension two linear section of X_5 containing a line L, we have*

$$\mathrm{Hom}_{X_5}(\mathcal{O}_L(-2), \mathcal{O}_{E_5}) \cong \mathbb{C}\langle\phi\rangle. \tag{6}$$

Furthermore,

(1) the non-zero map

$$\phi : \mathcal{O}_L(-2) \to \mathcal{O}_{E_5} \tag{7}$$

is injective and thus the cokernel of ϕ is isomorphic to coker$(\phi) = \mathcal{O}_{C_4}$ for a curve C_4 with Hilbert polynomial $\chi(\mathcal{O}_{C_4}(m)) = 4m + 1$ in X_5.

(2) The image of the map ϕ in (7) is isomorphic to im$(\phi) \cong \mathcal{O}_L(-1)$ if and only if C_4 is a non Cohen–Macaulay (CM)-curve.

Proof. Through some diagram chasing by using the structure sequence:

$$0 \to I_{E_5/X_5} \to \mathcal{O}_{X_5} \to \mathcal{O}_{E_5} \to 0, \tag{8}$$

we have $\mathrm{Hom}_{X_5}(\mathcal{O}_L(-2), \mathcal{O}_{E_5}) \cong \mathrm{Ext}^1_{X_5}(\mathcal{O}_L(-2), I_{E_5/X_5})$. Also, from the resolution (5) of I_{E_5/X_5}, there is an exact sequence

$$\begin{aligned} 0 = \mathrm{Ext}^1_{X_5}(\mathcal{O}_L(-2), \mathcal{O}_{X_5}(-1)^{\oplus 2}) &\to \mathrm{Ext}^1_{X_5}(\mathcal{O}_L(-2), I_{E_5/X_5}) \\ &\to \mathrm{Ext}^2_{X_5}(\mathcal{O}_L(-2), \mathcal{O}_{X_5}(-2)) = \mathbb{C} \to \cdots \end{aligned} \tag{9}$$

where the equality of the first term in (9) holds because of the Serre vanishing and the duality theorem, and the third one is

$$\mathrm{Ext}^2_{X_5}(\mathcal{O}_L(-2), \mathcal{O}_{X_5}(-2)) \cong \mathrm{Ext}^1_{X_5}(\mathcal{O}_{X_5}, \mathcal{O}_L(-2))^* \cong \mathrm{H}^1(\mathcal{O}_L(-2))^* \cong \mathrm{H}^0(\mathcal{O}_L).$$

For generic linear sections $\langle s_1, s_2\rangle$, the result (6) holds and thus it is true for any linear section.

(1) The kernel of the map ϕ is of the form $\ker(\phi) = \mathcal{O}_L(-k)$, $k \geq 3$, then im(ϕ) is a torsion subsheaf in $\mathcal{O}_{E_5}$. However, this contradicts the fact that E_5 is a CM-curve because E_5 is a locally complete intersection. Hence, $\ker(\phi) = 0$.

(2) If im$(\phi) = \mathcal{O}_L(-1)$, then the quotient is $\mathcal{O}_L(-1)/\mathcal{O}_L(-2) \cong \mathbb{C}_p \subset \mathcal{O}_{C_4}$ for a point $p \in C_4$. Hence, $\mathcal{O}_{C_4}$ has a torsion subsheaf. Conversely, let $\overline{C}$ be the maximal CM-subcurve of the non-CM curve C_4. Since X_5 does not contain any plane, the curve $\overline{C}$ has Hilbert polynomial $\chi(\mathcal{O}_{\overline{C}}(m)) = 4m + 1$ or $4m$ by the degree-genus formula (i.e., any non-planar CM-curve holds the genus bound $g \leq \frac{(d-2)(d-3)}{2}$). For the former case, $\overline{C} = C_4$, which is a contradiction to the assumption. Hence, $\chi(\mathcal{O}_{\overline{C}}(m)) = 4m$. In this case, by the purity of E_5, the kernel of the restriction map $\mathcal{O}_{E_5} \twoheadrightarrow \mathcal{O}_{\overline{C}}$ is a torsion-free sheaf supported on the line L. By comparing the Hilbert polynomials, we see that im$(\phi) = \mathcal{O}_L(-1)$. □

From the proof of (2) of Proposition 4.2, we need to know the non-existence of CM-curves in X_5 with Hilbert polynomial $4m$, which we will discuss in the next subsection.

4.2. **Elliptic quartic curves in X_5.** In this subsection, we prove the non-existence of elliptic quartic curves in X_5.

Proposition 4.3. *There does not exist* any *CM-curve E in X_5 with $\chi(\mathcal{O}_E(m)) = 4m$.*

Proof. Using (8), one can show that every elliptic quartic curve E on X_5 is contained in a hyperplane section of X_5. We now show that such curves do not exist. Let us assume that the CM-curve E is a complete intersection. Let $F = X_5 \cap H$ be a hyperplane section of $X_5 \subseteq \mathbb{P}^6$. According to [Pro91, Section 2], either F is a normal quintic del Pezzo surface with at most Du Val singularities, or F is a non-normal ruled surface. If F is normal, then let $\psi : \widetilde{F} \to F$ be the minimal resolution of F and let $H' = \psi^*\mathcal{O}_F(1)$. Note that $\widetilde{F}$ is a (weak) del Pezzo surface of degree 5. Suppose that $\widetilde{F}$ contains an elliptic curve E whose image in X_5 has degree 4, i.e., $H' \cdot E = 4$. By the adjunction formula, we have $E^2 = 4$ since $-K_{\widetilde{F}} \cdot E = H' \cdot E = 4$. This violates the Hodge index theorem ($16 = (H' \cdot E)^2 \geq (H' \cdot H')(E \cdot E) = 5 \times 4 = 20$). If F is non-normal, then, according to [Pro91, Lemma 2.1], there exists a morphism $\psi : \mathbb{F}_m \to F$, where $\mathbb{F}_m = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-m))$ is the m-th Hirzebruch surface for $m = 1$ or 3. We denote $H' = \psi^*\mathcal{O}_F(1)$, let s be the negative section, and let f be the fiber class of $\mathbb{F}_m$. Note that ψ is defined by a sublinear system of $|s + 3f|$ if $m = 1$, and of $|s + 4f|$ if $m = 3$. If $m = 1$, then $H' = s + 3f$ and $K_{\mathbb{F}_1} = -2s - 3f$. For a curve E with numerical class $E = as + bf$, the H'-degree $d(E)$ and the arithmetic genus $g(E)$ of E are given by

$$d(E) = 2a + b \quad \text{and} \quad g(E) = \frac{1}{2}(a-1)(2b-a-2).$$

Otherwise, if $m = 3$, then $H' = s + 4f$ and $K_{\mathbb{F}_3} = -2s - 5f$. For a curve E with numerical class $E = as + bf$, the H'-degree $d(E)$ and the arithmetic genus $g(E)$ of E are given by

$$d(E) = a + b \quad \text{and} \quad g(E) = \frac{1}{2}(a-1)(2b-3a-2).$$

In either case, there are no integers a and b such that $d(E) = 4$ and $g(E) = 1$. Lastly, if the CM-curve E is not a complete intersection, then its defining equations are given by two quadrics having the common linear factor and a cubic form (see [CCM16, Proposition 2.5, (2) and (3)]). This implies that a plane is contained in X_5, which is absurd.

□

4.3. **Construction of a morphism from a Grassmannian bundle to the Hilbert scheme.**

Proposition 4.4. *Let $C \subset X_5$ be a curve with Hilbert polynomial $\chi(\mathcal{O}_C(m)) = 4m + 1$. Then,*

(1) the curve C is CM and
(2) the linear span of the curve C is $\langle C \rangle = \mathbb{P}^4$.

Proof. (1) By the proof of (2) in Proposition 4.2, whenever the curve C is not a CM-curve, then its maximal CM-subcurve must be an elliptic quartic curve. Such curves do not exist in X_5 by Proposition 4.3.

(2) Since C is a CM-curve with $\chi(\mathcal{O}_C) = 1$, the structure sheaf $\mathcal{O}_C$ is stable by Lemma 3.17 in [CC11]. Clearly, $\langle C\rangle \not\subset \mathbb{P}^2$. Assume that $\langle C\rangle = \mathbb{P}^3$. In [CCM16, Theorem 6.1], it was proved that the sheaf $\mathcal{O}_C$ fits into an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus 3} \to \Omega_{\mathbb{P}^3}(-1) \to \mathcal{O}_S \to \mathcal{O}_C \to 0 \tag{10}$$

for some quadric surface $S \subset \mathbb{P}^3$. From the resolution of $\mathcal{O}_C$ in (10), one can see that the dimension $h^0(I_{C/\mathbb{P}^3}(2)) = 1$ for the ideal sheaf $I_{C/\mathbb{P}^3}$. Among the hypersurfaces defined by the five quadric generators of X_5 in (2), at least one intersection with $\langle C\rangle = \mathbb{P}^3$ yields a quadric surface. Since $h^0(I_{C/\mathbb{P}^3}(2)) = 1$, this surface is unique, and hence the del Pezzo variety X_5 contains the quadric surface S. With respect to the rank of the quadric form q defining S, one can easily see that it contradicts the geometry of X_5. In fact, if $\mathrm{rk}(q) = 4$, then $\mathbf{H}_1(S) \cong \mathbb{P}^1 \sqcup \mathbb{P}^1$ is contained in $\mathbf{H}_1(X_5) \cong \mathbb{P}^2$, which contradicts Bézout's theorem. If $\mathrm{rk}(q) = 3$, then there exists an infinite family of lines passing through the vertex, contradicting the fact that there are at most three lines passing through a point in X_5 ([Ili94, Section 2.1]). Finally, suppose that $\mathrm{rk}(q) \leq 2$. Then $\mathbb{P}^2 \subset S \subset X_5$, which is impossible. □

We are ready to propose one of the main ingredients in this paper. Let $\mathcal{L}$ be the universal lines in $\mathbf{H}_1(X_5) \times X_5$ with the projection map $\pi : \mathbf{H}_1(X_5) \times X_5 \to \mathbf{H}_1(X_5)$ into the first component. Let us consider the direct image sheaf $\mathcal{F} = \pi_* \mathcal{I}_{\mathcal{L}}(1)$. Since $\mathrm{H}^0(I_{L/X_5}(1)) \cong \mathbb{C}^5$ for each line L in X_5, the sheaf $\mathcal{F}$ is locally free on the Hilbert scheme $\mathbf{H}_1(X_5) \cong \mathbb{P}^2$ ([FN89, Section 2]). Let $p : \mathbf{G} = \mathrm{Gr}(2, \mathcal{F}) \to \mathbf{H}_1(X_5)$ be the relative Grassmannian bundle over $\mathbf{H}_1(X_5)$ which parametrizes linear subspaces $\mathbb{P}^4$ containing a line in X_5. As a set, the Grassmannian bundle $\mathbf{G}$ is an incident variety:

$$\mathbf{G} = \{(l, H) \mid l \subset H, H \cong \mathbb{P}^4 \subset \mathbb{P}^6, [l] \in \mathbf{H}_1(X_5)\}.$$

Proposition 4.5. *Under the above definition and notation, there exists an injective morphism from the flag variety $\mathbf{G}$ to the Hilbert scheme $\mathbf{H}_4(X_5)$ which associates (l, H) to the support of the cokernel coker$\{\mathcal{O}_L(-2) \stackrel{\phi}{\to} \mathcal{O}_{X_5 \cap H}\}$ (Section 4.1).*

Proof. Let $\mathcal{U}$ be the universal subbundle on the relative Grassmannian $\mathbf{G}$. We relativize the exact sequence in (5). Let $\overline{\pi} : \mathbf{G} \times X_5 \to \mathbf{G}$ be the projection map into the first component.

$$\begin{tikzcd}
\mathbf{G} \times X_5 \arrow[r, "p\times \mathrm{id}"] \arrow[d, "\overline{\pi}"'] & \mathbf{H}_1(X_5) \times X_5 \arrow[r] \arrow[d, "\pi"] & X_5 \\
\mathbf{G} \arrow[r, "p"] & \mathbf{H}_1(X_5) &
\end{tikzcd}$$

From the natural map $\pi^*\pi_*\mathcal{I}_{\mathcal{L}}(1) \to \mathcal{I}_{\mathcal{L}}(1)$ and by pulling back the canonincal subbundle $\mathcal{U} \subset p^*\mathcal{F}$ on the Grassmannian bundle $\mathbf{G}$ along the map $\overline{\pi}$, we have a sheaf homomorphism

$$\Phi : \overline{\pi}^*\mathcal{U} \to \overline{\pi}^*p^*\mathcal{F} = (p\times \mathrm{id})^*\pi^*\mathcal{F} = (p\times \mathrm{id})^*\pi^*\pi_*\mathcal{I}_{\mathcal{L}}(1) \to (p\times \mathrm{id})^*\mathcal{I}_{\mathcal{L}}(1) \subset \mathcal{O}(1).$$

The Koszul complex of $\overline{\pi}^*\mathcal{U}(-1) := \overline{\pi}^*\mathcal{U}\otimes \mathcal{O}_{X_5}(-1)$ with the twisted map Φ provides a locally free resolution of elliptic quintic curves in X_5. That is, there exists a locally free resolution:

$$0 \to \wedge^2\overline{\pi}^*\mathcal{U}(-2) \to \overline{\pi}^*\mathcal{U}(-1) \to \mathcal{I}_{\mathcal{E}_5} \to 0$$

where $\mathcal{E}_5$ is the universal elliptic quintic curve over $\mathbf{G}\times X_5$. Also, we can glue maps in (7) as follows. Consider the relative Hom-sheaf

$$\mathcal{M} = \mathcal{H}om_{\overline{\pi}}((p\times \mathrm{id})^*\mathcal{O}_{\mathcal{L}}(-2), \mathcal{O}_{\mathcal{E}_5})$$

on $\mathbf{G}$. As we have seen in the proof of Proposition 4.2, each fiber of $\mathcal{M}$ is completely determined by the structure sheaf of a line. Therefore, $\mathcal{M} \cong p^*\mathcal{O}_{\mathbb{P}^2}(k)$, $k\in\mathbb{Z}$ for the canonical projection map $p:\mathbf{G}\to \mathbf{H}_1(X_5)\cong \mathbb{P}^2$. From the adjunction formula, there exists an isomorphism:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{G}}(\mathcal{M},\mathcal{M}) &= \mathrm{Hom}_{\mathbf{G}}(\mathcal{O}_{\mathbf{G}}, \mathcal{M}\otimes\mathcal{M}^*) = \mathrm{H}^0(\mathbf{G}, \mathcal{H}om_{\overline{\pi}}((p\times \mathrm{id})^*\mathcal{O}_{\mathcal{L}}(-2), \mathcal{O}_{\mathcal{E}_5}\boxtimes\mathcal{M}^*)) \\ &\cong \mathrm{Hom}_{\mathbf{G}\times X_5}((p\times \mathrm{id})^*\mathcal{O}_{\mathcal{L}}(-2), \mathcal{O}_{\mathcal{E}_5}\boxtimes\mathcal{M}^*).\end{aligned} \tag{11}$$

The identity map id $\in \mathrm{Hom}(\mathcal{M},\mathcal{M})$ in (11) induces a sheaf homomorphism

$$\Psi : (p\times \mathrm{id})^*\mathcal{O}_{\mathcal{L}}(-2)\boxtimes\mathcal{O}_{\mathbb{P}^2}(k) \to \mathcal{O}_{\mathcal{E}_5}$$

over $\mathbf{G}\times X_5$. The cokernel of Ψ parametrizes rational quartic curves in X_5. Hence, from the universal property of the Hilbert scheme, we have a morphism

$$\overline{\Psi} : \mathbf{G} \longrightarrow \mathbf{H}_4(X_5). \tag{12}$$

For the injectivity of $\overline{\Psi}$, suppose that $C_4 = C_4'$. Then since $\langle C_4\rangle = \langle C_4'\rangle$, they determine the same line. $\square$

5. Geometry of Hilbert scheme of rational quartic curves

In this section, we will prove that $\mathbf{H}_4(X_5)$ is smooth (Proposition 5.3) and irreducible (Proposition 5.7). Hence, we conclude that the injective morphism $\overline{\Psi}$ in (12) is an isomorphism by Zariski's main theorem.

5.1. $\mathbb{C}^*$-fixed rational quartic curves.

Lemma 5.1. *Let* $\mathbf{G}$ *be the Grassmannian bundle over* $\mathbf{H}_1(X_5) \cong \mathbb{P}^2$*. Then the fixed loci of* $\mathbf{G}$ *are isolated. That is, it consists of* 30 *points.*

Proof. Let C be $\mathbb{C}^*$-fixed rational quartic curves in X_5. From Proposition 4.4, $\langle C\rangle = \mathbb{P}^4$. So the linear space $\mathbb{P}^4$ is fixed. Furthermore, $X_5 \cap \mathbb{P}^4 = C \cup L$ is also fixed. Therefore, we need to consider only fixed lines in X_5. The fixed lines are defined by the ideal in $\mathbb{P}^6$ (Section 3.2):

$$l_0 = \langle a_6, a_2, a_0, a_{-2}, a_{-6}\rangle,\ l_1 = \langle a_2, a_0, a_{-2}, a_{-4}, a_{-6}\rangle,\ l_2 = \langle a_{-2}, a_0, a_2, a_4, a_6\rangle.$$

The fiber of $\mathbf{G}$ over each line l_i is the Grassmannian $\mathrm{Gr}(2, l_0)$ when we regard l_i as the subspace of $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^6}(1))$. If we count the multiplicity of weights in the space $\wedge^2 l_0$, one can easily see that every fixed locus is isolated by Lemma 3.1. The defining ideals are presented in Table 2. Among the total of thirty cases, *symmetric* ones are listed only once — only the second case (denoted with *) has no symmetry. Here, we mean symmetric if the defining ideals have the same generators after replacing a_i with a_{-i}. □

TABLE 2. $\mathbb{C}^*$-fixed reducible quartic curves

No.	$I_{C/\mathbb{P}^6}$	C
1	$\langle a_0, a_4, a_{-2}a_{-4}, a_2a_{-4}, a_6a_{-4}, 3a_{-2}^2 + a_2a_{-6}, 9a_2a_{-2} - a_6a_{-6}, 3a_2^2 + a_6a_{-2}\rangle$	$l_2 + C_3$
2*	$\langle a_{-2}, a_2, a_4a_{-6}, a_0a_{-4}, a_4a_{-4}, a_6a_{-4}, 8a_0^2 + a_6a_{-6}, a_4a_0\rangle$	$l_1 + l_2 + C_2$
3	$\langle a_{-2}, a_2, a_4a_{-6}, a_0a_{-4}, a_6a_{-4}, 8a_0^2 + a_6a_{-6}, a_4a_0, a_6a_4\rangle$	$l_0 + l_2 + C_2$
4	$\langle a_2, a_4, a_0a_{-4}, a_6a_{-4}, a_{-2}^2, a_0a_{-2}, a_6a_{-2}, 8a_0^2 + a_6a_{-6}\rangle$	$2l_2 + C_2$
5	$\langle a_0, a_2, a_4a_{-6}, a_6a_{-6}, a_6a_{-4}, a_{-2}^2, a_4a_{-2}, a_6a_{-2}\rangle$	$l_0 + l_1 + 2l_2$
6	$\langle a_0, a_2, a_{-2}a_{-6}, a_4a_{-6}, a_6a_{-6}, a_{-2}^2, 3a_4a_{-2} - a_6a_{-4}, a_6a_{-2}\rangle$	$2l_0 + l_1 + l_2$
7	$\langle a_{-2}, a_4, a_{-4}^2, 4a_0a_{-4} - a_2a_{-6}, a_2a_{-4}, 8a_0^2 + a_6a_{-6}, 2a_2a_0 + a_6a_{-4}, a_2^2\rangle$	$2C_2$
8	$\langle a_{-2}, a_6, 4a_0a_{-4} - a_2a_{-6}, 3a_2a_{-4} - a_4a_{-6}, a_0^2, a_2a_0, a_4a_0, a_2^2\rangle$	$2l_0 + 2l_2$
9	$\langle a_0, a_2, a_4a_{-6}, a_6a_{-6}, a_{-2}^2, 3a_4a_{-2} - a_6a_{-4}, a_6a_{-2}, a_6^2\rangle$	$2l_0 + 2l_2$
10	$\langle a_0, a_6, a_2a_{-6}, 3a_2a_{-4} - a_4a_{-6}, a_{-2}^2, a_2a_{-2}, a_4a_{-2}, a_2^2\rangle$	$2l_0 + 2l_2$
11	$\langle a_{-2}, a_6, a_0a_{-6}, 4a_0a_{-4} - a_2a_{-6}, 3a_2a_{-4} - a_4a_{-6}, a_0^2, a_2a_0, 3a_2^2 - 4a_4a_0\rangle$	$3l_0 + l_2$
12	$\langle a_2, a_6, a_4a_{-6}, 3a_{-2}^2 - 4a_0a_{-4}, a_0a_{-2}, a_4a_{-2}, a_0^2, a_4a_0\rangle$	$l_0 + 3l_2$
13	$\langle a_0, a_6, 3a_2a_{-4} - a_4a_{-6}, 3a_{-2}^2 + a_2a_{-6}, a_2a_{-2}, a_4a_{-2}, a_2^2, a_4a_2\rangle$	$l_0 + 3l_2$
14	$\langle a_4, a_6, 3a_{-2}^2 - 4a_0a_{-4} + a_2a_{-6}, 2a_0a_{-2} - 3a_2a_{-4}, a_2a_{-2}, a_0^2, a_2a_0, a_2^2\rangle$	$4l_2$
15	$\langle a_2, a_6, 3a_{-2}^2 - 4a_0a_{-4}, 2a_0a_{-2} + a_4a_{-6}, a_4a_{-2}, a_0^2, a_4a_0, a_4^2\rangle$	$4l_2$

Example 5.2. There exists a unique $\mathbb{C}^*$-fixed rational normal curve C_4 whose defining equation is given by

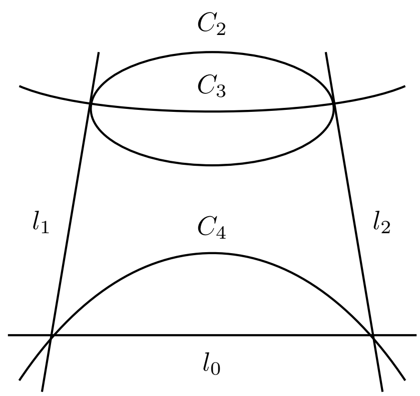
$$I_{C_4/\mathbb{P}^6} = \langle a_{-6}, a_6, 3a_{-2}^2 - 4a_0a_{-4}, 2a_0a_{-2} - 3a_2a_{-4}, a_2a_{-2} - 2a_4a_{-4}, \\ 4a_0^2 - 9a_4a_{-4}, 2a_2a_0 - 3a_4a_{-2}, 3a_2^2 - 4a_4a_0\rangle,$$

where the degree two parts of the ideal $I_{C_4/\mathbb{P}^6}$ are given by 2×2-minors of the matrix

$$\begin{bmatrix} \frac{27}{16}a_4 & \frac{9}{8}a_2 & a_0 & a_{-2} \\ \frac{9}{8}a_2 & a_0 & a_{-2} & \frac{4}{3}a_{-4} \end{bmatrix}.$$

In Section 3, we have already analyzed all of $\mathbb{C}^*$-fixed rational curves of degree ≤ 3. Hence, by Example 5.2, the reducible fixed curve up to the degree $d \leq 4$ has the configurations as in Figure 2.

FIGURE 2. Reducible rational curves of degree $d \leq 4$



Proposition 5.3. *The Hilbert scheme $\mathbf{H}_4(X_5)$ is a smooth variety.*

Proof. For each $\mathbb{C}^*$-fixed curve C in Table 2, one can check that the dimension of the deformation space at C is 8-dimensional by using `Macaulay2` ([GS]). For the code used in our computation, see Example 5.4 below. Also, in the next subsection, we will prove the irreducibility of $\mathbf{H}_4(X_5)$. Hence, we finish the proof. □

Example 5.4. The dimension of the deformation space at the multiplicity 4-line (No. 15 in Table 2) can be computed using the following `Macaulay2` code.

```
R = QQ[a6,a4,a2,a0,am2,am4,am6];
--- Negative weights are denoted by 'm'
I = ideal(a6*am2 - 4*a4*a0 + 3*a2^2, a6*am4 - 3*a4*am2 + 2*a2*a0,
a6*am6 - 9*a2*am2 + 8*a0^2, a4*am6 - 3*a2*am4 +2*a0*am2, a2*am6 - 4*a0*am4 +
    3*am2^2);
--- The defining ideal of the quintic del Pezzo threefold
J = ideal(a2, a6, 3*am2^2 - 4*a0*am4, 2*a0*am2 + a4*am6, a4*am2, a0^2, a4*a0,
    a4^2);
```

```
--- defining ideal of the multiplicity 4 line
numcols basis(0,Hom(J/I,R/J))
--- The dimension of the deformation space at R/J
```

Remark 5.5. For a pair $(l, \mathbb{P}^4)$, $l \subset \mathbb{P}^4$ we have unique rational quartic curve C. Let us assume that there exists a hyperplane $H(\cong \mathbb{P}^5)$ which contains $\mathbb{P}^4$ and the hyperplane section $H \cap X$ is a smooth del Pezzo surface. Then from the spectral sequence [CKK25, Lemma 4.16], one can easily check that the Hilbert scheme $\mathbf{H}_4(X)$ is smooth at the closed point $[C]$.

5.2. **Irreducibility of the Hilbert scheme.** The locally free resolutions of rational curves of lower degrees in X_5 were presented in [San14]. Let $\mathcal{U}$ (resp. $\mathcal{Q}$) be the universal sub(resp. quotient)-bundle on $\mathrm{Gr}(2, V_5)$. We use the same notation to denote its restriction to X_5.

Proposition 5.6 ([San14, Proposition 2.20, 2.32 and 2.46]). *Under the above definition and notation, for each rational curve C_d with degree $\deg(C_d) = d \leq 3$, the locally free resolution of an ideal sheaf I_{C_d/X_5} is given by*

(1) $(d = 1)$ $0 \to \mathcal{U} \to \mathcal{Q}^* \to I_{C_1/X_5} \to 0$,
(2) $(d = 2)$ $0 \to \mathcal{O}_{X_5}(-1) \to \mathcal{U} \to I_{C_2/X_5} \to 0$,
(3) $(d = 3)$ $0 \to \mathcal{O}_{X_5}(-1)^{\oplus 2} \to \mathcal{Q}(-1) \to I_{C_3/X_5} \to 0$.

In particular, the resolution of a conic C_2 is a regular section of the dual bundle $\mathcal{U}^*$ and thus its normal bundle in X_5 is isomorphic to $N_{C_2/X_5} \cong \mathcal{U}^*|_{C_2}$ ([San14, Corollary 2.33]).

Proposition 5.7. *The Hilbert scheme $\mathbf{H}_4(X_5)$ of rational quartic curves in X_5 is irreducible.*

Proof. Let $[C] \in \mathbf{H}_4(X_5)$. We denote by C_{red} the reduced scheme of the curve C. Note that the expected dimension of the Hilbert scheme $\mathbf{H}_4(X_5)$ at $[C]$ is

$$\dim \mathrm{Ext}^0(I_{C/X_5}, \mathcal{O}_C) - \dim \mathrm{Ext}^1(I_{C/X_5}, \mathcal{O}_C) = 8$$

by the stability of the structure sheaf $\mathcal{O}_C$ ([CC11, Lemma 3.17]) and $K_{X_5} = -2H$. Hence, each connected component of the Hilbert scheme $\mathbf{H}_4(X_5)$ containing the curve C has dimension at least 8 ([Kol96, Theorem I.2.15]). Since the space of reduced rational quartic curves in X_5 is irreducible by [LT19, Theorem 7.9], it is enough to prove that each locus of reducible curves with fixed types of a reduced curve has dimensions at most 7. Note that the boundary of the Hilbert scheme $\mathbf{H}_3(X_5) \cong \mathrm{Gr}(2, 5)$ is 5-dimensional. We will estimate the upper bound of the dimension of multiple structures over a fixed reduced curve. For the detailed description of the multiple structure of a CM-curve in smooth projective threefolds, see the papers [BF86, Nol97, NS03].

Case 1. $\deg(C_{\text{red}}) = 3$. Let us denote by $D = C_{\text{red}}$. If $p_a(D) \neq 0$, then D is a planar curve with degree 3, which violates the fact that X_5 does not contain any plane. Hence, $p_a(D) =$

0 and thus structure sequence is given by

$$0 \to I_{D/C} \to \mathcal{O}_C \to \mathcal{O}_D \to 0. \tag{13}$$

Clearly, the Hilbert polynomial of the ideal sheaf $I_{D/C}$ is $\chi(I_{D/C}(m)) = m$. Hence, it is supported on a line $L \subset C$. However, C is a CM-curve and thus $I_{D/C} \cong \mathcal{O}_L(-1)$. On the other hand, the non-split extensions of (13) are parametrized by

$$\mathrm{Ext}^1(\mathcal{O}_D, \mathcal{O}_L(-1)) \cong \mathrm{Hom}(I_{D/X_5}, \mathcal{O}_L(-1))$$

up to the scalar multiplication (cf. [HL10, Example 2.1.12]). However, one can easily see that

$$\dim \mathrm{Hom}(I_{D/X_5}, \mathcal{O}_L(-1)) \leq 2$$

from the locally free resolution of I_{D/X_5} in Proposition 5.6. Hence, the locus of non-split extensions of (13) is at most 1-dimensional.

Case 2. $\deg(C_{\text{red}}) = 2$. The possible support of C_{red} is a smooth conic or a pair $L_1 \cup L_2$ $L_1 \neq L_2$ of lines meeting at a point.

Case 2-1. $[C] = 2[Q]$ for some smooth conic $Q \subset X_5$. Such a locus is at most 1-dimensional. The details are as follows. From the inclusion $Q \subset C$, there is a structure sequence

$$0 \to \mathcal{O}_Q(-p) \to \mathcal{O}_C \to \mathcal{O}_Q \to 0 \tag{14}$$

for some $p \in Q$. Note that the line bundle $\mathcal{O}_Q(-p)$ on Q is isomorphic to each other for all $p \in Q$. Conversely, the long exact sequence ([CKK25, Lemma 4.16])

$$0 = \mathrm{Ext}^1_Q(\mathcal{O}_Q, \mathcal{O}_Q(-p)) \to \mathrm{Ext}^1_{X_5}(\mathcal{O}_Q, \mathcal{O}_Q(-p)) \to \mathrm{H}^0(N_{Q/X_5} \otimes O_Q(-p)) \cong \mathbb{C}^2 \to \cdots$$

says that non-split extensions in (14) are parametrized by $\mathbb{P}^1$ or a point because $N_{Q/X_5} \cong \mathcal{U}^*|_Q$ (Proposition 5.6). Hence, we proved the claim.

Case 2-2. $[C] = 2[L_1] + 2[L_2]$, $L_1 \neq L_2$ for some pair of lines L_1 and L_2 meeting at a point. The double structure of $D = L_1 \cup L_2$ is at most 3-dimensional. In detail, since D is a locally complete intersection, a double structure on the curve D is determined by a surjective morphism

$$\psi : N^*_{D/X_5} = I_{D/X_5}/I^2_{D/X_5} \twoheadrightarrow \mathcal{L} \tag{15}$$

for a line bundle $\mathcal{L}$ on D with $\chi(\mathcal{L}(m)) = 2m$ ([BF86, Section 1]). As tensoring the line bundle $\mathcal{L}$ into the structure sequence $0 \to \mathcal{O}_{L_2}(-1) \to \mathcal{O}_D \to \mathcal{O}_{L_1} \to 0$, we have

$$0 \to \mathcal{O}_{L_2}(-a-2) \to \mathcal{L} \to \mathcal{O}_{L_1}(a) \to 0, \ a \in \mathbb{Z}.$$

From the fact that $N_{D/X_5} \cong \mathcal{U}^*|_D$ (Proposition 5.6), $\mathrm{Hom}(N^*_{D/X_5}, \mathcal{O}_{L_i}(k)) \neq 0$ if and only if $k \geq -1$. From this, one can check whenever ψ is surjective, $a = -1$ or 0. If $a = -1$, then $\mathrm{Hom}(N^*_{D/X_5}, \mathcal{O}_{L_i}(-1)) \cong \mathbb{C}$ because of $N_{D/X_5} \cong \mathcal{U}^*|_D$ again. Hence in this case, $\dim \mathrm{Hom}(N^*_{D/X_5}, \mathcal{L}) \leq 2$. Note that if we replace the role of the lines L_1 and L_2, we may obtain another family of line bundles $\mathcal{L}$. On the other hand, by upper semicontinuity and $\dim T_{[L_1]}\mathbf{H}_1(X_5)=2$ ([FN89, Section 2]), we have

$$\dim \mathrm{Ext}^1(\mathcal{O}_{L_1},\mathcal{O}_{L_2})\leq \dim \mathrm{Ext}^1(\mathcal{O}_{L_1},\mathcal{O}_{L_1})=2.$$

Hence, such a line bundle $\mathcal{L}$ is at most 1-dimensional. After all, considering the case $a=0$ as well, the space of surjective morphisms ϕ in (15) is at most 3-dimensional.

Case 3. Let $[C]=3[L_1]+[L_2]$, $L_1\neq L_2$ for a pair of lines L_1 and L_2 meeting at a point. Such a locus is at most 4-dimensional. Let us consider degree 3 maximal CM-subcurve $C_3\subset C$ supported on L_1 which is obtained from the ideal quotient $(I_{C/X_5}:I_{L_2/X_5})$. Since the length of the intersection $C_3\cap L_2$ is $1\leq l(C_3\cap L_2)\leq 3$, the Hilbert polynomial of the curve C_3 is $\chi(\mathcal{O}_{C_3}(m))=3m+k$, $1\leq k\leq 3$ (cf. [NS03, Section 5]). Also, the possible CM-filtrations of the curve C_3 are of types (cf. [Nol97, Section 2]):

(a) $(m+1,2m+1,3m+1)$,
(b) $(m+1,2m+1,3m+2)$,
(c) $(m+1,2m+1,3m+3)$,
(d) $(m+1,2m+2,3m+3)$,

where each component of the 3-tuple indicates the Hilbert polynomial of sub-curves in the filtration of C_3. Note that a planar double line (i.e., Hilbert polynomial $2m+1$) exists only if the line is non-free ([FN89, Section 1]) and the non-planar double lines (i.e., Hilbert polynomial $2m+2$) are parametrized by the projectivization of the tangent space of a line in X_5.

For the case (a), there is an exact sequence

$$0\to\mathcal{O}_{L_2}(-1)\to\mathcal{O}_C\to\mathcal{O}_{C_3}\to 0,$$

where $\chi(\mathcal{O}_{C_3}(m))=3m+1$ and thus C is determined by the same method as in Case 1. Note that the multiple curve C_3 supported on the non-free line L_1 is unique by [Chu23, Lemma 4.2]. Hence, the dimension of such a locus is ≤ 1.

For the case (b) by diagram chasing, there is an exact sequence $0\to F\to\mathcal{O}_C\to\mathcal{O}_{C_2}\to 0$ where F fits into the extension $0\to\mathcal{O}_{L_2}(-2)\to F\to\mathcal{O}_{L_1}\to 0$ and C_2 is a curve supported on L_1 with Hilbert polynomial $\chi(\mathcal{O}_{C_2}(m))=2m+1$. In terms of ideal sheaves, we have an exact sequence $0\to I_{C/X_5}\to I_{C_2/X_5}\to F\to 0$. However, the map $I_{C_2/X_5}\to F$ is at most 3-dimensional because of $\mathrm{Hom}(I_{C_2/X},\mathcal{O}_{L_2}(-2))=0$ (Proposition 5.6). Hence, such a non-trivial extension is at most $3+3=6$-dimensional. The same reasoning holds for the case (c).

For the case (d) by diagram chasing again, there exists an exact sequence

$$0\to G\to\mathcal{O}_C\to\mathcal{O}_{C_2}\to 0,$$

where G fits into the exact sequence $0 \to \mathcal{O}_{L_2}(-3) \to G \to \mathcal{O}_{L_1} \to 0$ with $\chi(\mathcal{O}_{C_2}(m)) = 2m+2$. In terms of ideal sheaves, we have an exact sequence

$$0 \to I_{C/X_5} \to I_{C_2/X_5} \to G \to 0. \tag{16}$$

Conversely, we estimate the dimension of $\mathrm{Hom}(I_{C_2}, G)$. From the structure ideal sequence $0 \to I_{C_2/X_5} \to I_{L_1/X_5} \to \mathcal{O}_{L_1} \to 0$, one can easily see that

$$\dim \mathrm{Hom}(I_{C_2/X_5}, \mathcal{O}_{L_1}) \leq 2 \text{ and } \dim \mathrm{Hom}(I_{C_2/X_5}, \mathcal{O}_{L_2}(-3)) \leq 1.$$

Therefore, the dimension of C's in (16) is at most

$$\dim \mathrm{Hom}(I_{C_2/X_5}, G) + \dim \mathbb{P}\mathrm{Ext}^1(\mathcal{O}_{L_1}, \mathcal{O}_{L_1}) \leq (2+1)+1 = 4.$$

Case 4. $\deg(C_{\text{red}}) = 1$. Clearly, C_{red} is a line. There exists a one-dimensional family of quadruple structure on a non-free line in X_5. This case is completely analogous to the first paragraph of the proof in [CKK25, Proposition 4.14]. So we skip the proof. □

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